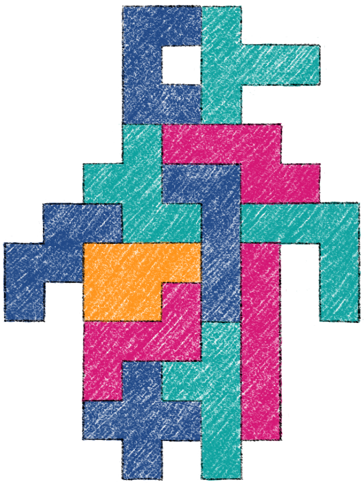


About Problem Solving

Many years ago, I attended a seminar run by the renowned mathematics educator Charles Lovitt. It was an interactive and eye-opening experience. Charles presented a series of activities to a room full of mathematics teachers, and we all attempted the activities as students in his class.

There was one activity that really gave me pause. Called "Win At The Fair", it positions students as fundraisers: they were tasked with running a game to raise money at the school fair. Participants in this game would pay $1 to play, and ultimately receive a monetary prize, the size of which was dependent on chance. As students, we played several rounds of this game to make sure we understood how it worked, while Charles managed a running tally of revenue and prizes on the whiteboard.

The punch line was that, statistically, the game would lose money: the prizes would exceed the revenue, and that was exactly what happened. Questions arising at this point were very revealing. Given that we're trying to raise money for the school, is this a good game to run? How much should the game really pay out in prizes? How might we modify the game so that we reduce the average prize to an acceptable fraction of the revenue?

There is no question in my mind that this game provides a powerful learning experience for students of all ages. The learning is certainly mathematical, but also something more: it teaches about life, and incentives, and how to think outside the box. I think all of us learned something useful that day. Since then, I have run this activity many times, with different cohorts of students. In doing so, I expect them to develop a realisation that gambling games are rigged. The house always wins.

What I find particularly fascinating about this activity, however, is that I cannot think of a way that we can reasonably assess the learning. What could we possibly ask, to find out if the students had actually developed depth of understanding? We could just as well save time by providing the students with a lecture on this topic - but I would suggest that, if we taught this concept as a lecture, then the learning that occurs is going to be qualitatively different. The trouble is that, when we examine students' understanding, we are unlikely to be able to tell whether they had been told how to answer, or if they really, truly, understood the implications in such a situation. With our limited assessment capabilities, it can be very difficult to ascertain the quality of students' thinking[1].

Learning through Problem Solving

As noted in Volume 1, problem solving is necessarily associated with some form of struggle on the part of the learner. I can sympathise with teachers who are reluctant to teach mathematics through problem solving: in comparison to direct instruction, problem solving is a complicated and messy business. There are a multitude of reasons why the learning in a problem solving lesson might not progress as the teacher had planned.

Problem solving lessons share some of the characteristics of "Win At The Fair". In both instances, students are being taken on a journey. It is guided, in the sense that we are providing them with a context with which we expect them to interact, and there is an outcome that, if achieved, might indicate that the learning has been successful. Like "Win At The Fair", it can be very difficult to assess what the students might have learned. There is a human element to the learning that is not reducible to a quantifiable statement.

While problem solving is not necessarily the most efficient way to get through content, genuine problem solving opportunities are needed for students to develop a critical subset of mathematical skills. These skills may, or may not, be prioritised

1. Skemp, R.R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77*, pp. 20-26.

in day-to-day mathematics assessments[2], but the problem solving activities have value beyond what is immediately assessable. They provide students with the opportunity to draw upon their own resources, and engage in learning that requires them to think[3].

About Mathematics Assessment

Mathematics assessments, and indeed mathematics competitions, are generally understood to prioritise speed and accuracy. However, outside of a competitive environment, it is worth considering whether these aspects of mathematical attainment comprise the entirety, or even the majority, of what we value in students' mathematical development. Why, indeed, do we learn mathematics? What is it that we, as a society, are trying to achieve?

While accuracy (perhaps more so than speed) is certainly an important consideration for allowing students to demonstrate what they can do, if we take a longer view, there are aspects to mathematics learning that are at least as important as accuracy. A correct answer is a symptom - perhaps a revealing symptom - of proficiency, with regard to a particular mathematical question. Understanding the solution "all the way to the bottom[4]" is something else entirely. If we are honest with ourselves about what our students truly understand, then we will likely find that there are procedures that the students have learned to use, but don't know well enough to explain why they work.

This phenomenon has been noted for some time. As mathematics educators, we need to consider the amount of time and effort required to guide our students towards the kind of understanding that is characterised by "knowing both what to do and why[1]". For short term results, knowing how to apply formulas and procedures is certainly useful. However, understanding why a procedure works, and being able to use it flexibly when the context changes or the question is asked in a different way, gives us clarity and insight that can inform our approach when considering a more diverse range of problems. This understanding makes the mathematics easier to remember, even if it is initially harder to learn[1].

Demonstrating Mathematical Capability

It is notable that problem solving capability is one of the characteristics that distinguish high levels of mathematical attainment. In NSW, performance bands for mathematics assessments in Year 12 (matriculation) include descriptors for students who exhibit different levels of achievement. Every mathematics course, ranging from Mathematics Standard 1 through to Mathematics Extension 2, explicitly notes the centrality of problem solving for indicating high levels of mathematical capability.

For example, for Mathematics Standard 1:

- selects and uses a variety of strategies to solve mathematical problems[5]

For Mathematics Extension 2:

- creatively synthesises mathematical concepts, techniques and results to efficiently solve problems
- demonstrates sophisticated multi-step logic and mathematical insight to solve problems and prove results in familiar and unfamiliar situations
- constructs, analyses and applies mathematical models effectively to solve problems in a wide variety of situations[6]

It is worth noting that these descriptors, as well as the top band descriptors for all of the mathematics courses in between, have relevance irrespective of individual perspectives regarding how students best learn mathematics[7]. Problem solving skills are foundational for "doing" mathematics. The confidence to apply these skills in a variety of situations, mathematical or otherwise, is an asset for all students.

2. Smith, G., Wood, L., Coupland, M., Stephenson, B., Crawford, K. & Ball, G. (1996). Constructing mathematical examinations to assess a range of knowledge and skills. *International Journal of Mathematical Education in Science and Technology, 27*(1), pp. 65-77.

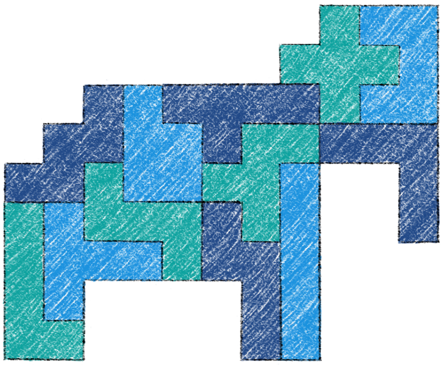
3. Liljedahl, P. (2020). Building thinking classrooms in mathematics, Grades K-12: 14 teaching practices for enhancing learning.

4. Ellenberg, J. (2015). How not to be wrong: The power of mathematical thinking.

5. NSW Education Standards Authority [NESA] (2024). Mathematics Standard 11-12 Syllabus (2024). https://curriculum.nsw.edu.au/learning-areas/mathematics/mathematics-standard-11-12-2024/assessment

6. NSW Education Standards Authority [NESA] (2024). Mathematics Extension 2 11-12 Syllabus (2024). https://curriculum.nsw.edu.au/learning-areas/mathematics/mathematics-extension-2-11-12-2024/assessment

7. Munter, C., Stein, M.K. & Smith, M.S. (2015). Dialogic and direct instruction: Two distinct models of mathematics instruction and the debate(s) surrounding them. *Teachers College Record, 117*.



The Strategies

Mathematics is a creative subject requiring abstract thought. Students naturally reason and use creative strategies when they seek patterns and relationships that will enable them to solve challenging unfamiliar problems. The generalisations they make can then be used to solve problems with the same mathematical structure.

Through the process of problem solving and class discussion of the strategies used, students will also develop skills they can use when faced with more unfamiliar problems, to be able to:

- Describe and represent mathematical situations in a variety of ways
- Select and apply appropriate problem-solving strategies in undertaking investigations
- Give valid reasons for supporting one possible solution over another.

Problems can often be solved in many different ways. For this reason, different methods of solution will be suggested for each problem, with particular emphasis on:

Guess, Check and Refine

With this strategy, the student makes a reasonable guess of the answer, and then checks the guess against the conditions of the problem. If the first guess is not correct, the student obtains more information that may lead to the answer.

Beginners in particular are urged to use “Guess, Check and Refine” often, until they catch the “feel” of solving problems.

Draw a Diagram

If a problem is not illustrated, sometimes it is helpful for the student to draw a diagram.

A picture may reveal information that may not be obvious just by reading the problem.

It is also useful for keeping track of where the student is up to in a multi-step problem.

Find a Pattern

One of the most frequently used problem solving strategies is that of recognising and extending a pattern.

Students can often simplify a difficult problem by identifying a pattern in it, and then applying that pattern to the problem situation.

Build a Table

A table displays information so that it is easily located and understood, and missing information becomes obvious.

If students are not given the data for a problem, and must generate it themselves, a table is an excellent way to record what they have done so they don't have to repeat their efforts.

A table can also be invaluable for detecting significant patterns.

Work Backwards

If a problem describes a procedure and then specifies the final result, this method usually makes the problem much easier to solve.

Make an Organised List

Listing every possibility in an organised way is an important tool.

How students organise the data often reveals additional information.

Solve a Simpler Related Problem

Many hard problems are actually relatively straightforward problems that have been extended to larger numbers.

Replacing the large numbers with smaller numbers can introduce patterns that allow insights into how to solve the original problem.

Eliminate All But One Possibility

Deciding what a quantity is not, can narrow the field to a very few possibilities.

These can then be tested against the conditions of the original problem.

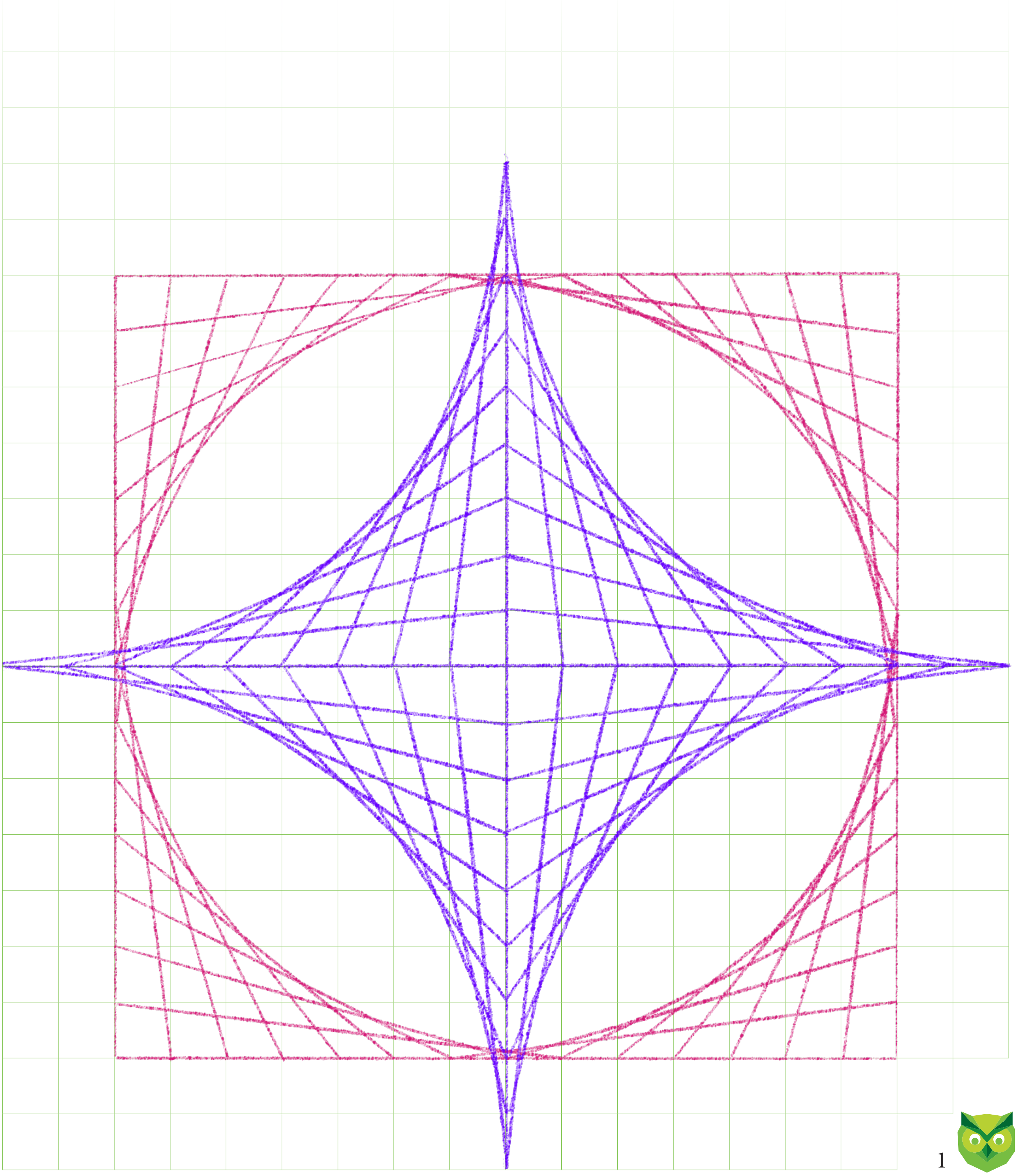
Convert to a More Convenient Form

There are times when changing some of the conditions of a problem makes a solution clearer or more convenient.

Divide a Complex Shape

Sometimes it is possible to divide an unusual shape into two or more common shapes that are easier to work with.

Problem Set 1



Problem Set 1

1 Peter made marmalade sandwiches and vegemite sandwiches.

He made 8 more marmalade sandwiches than vegemite sandwiches.

He made 30 sandwiches in total.

How many marmalade sandwiches did Peter make?

2 Martin, Nick, Olive and Penny are waiting in a line.

Nick is between Olive and Martin.

Martin is next to Penny.

If Penny is third in the line, who is first in line?

3 Suppose the time is now 5 o'clock on a twelve-hour clock.

What time will this clock show 125 hours from now?

4 I am making a pattern out of equilateral triangles.

Each triangle has a perimeter of 3 cm.

The diagrams show what the pattern looks like with 1, 2, 3, and 4 triangles.

Find the perimeter, in centimetres, for a pattern made up of 10 triangles.

5 In the subtraction shown, letters are used to represent digits.

Different letters do NOT necessarily represent different digits.

What is the four-digit number represented by ***ABCD***?

$$\begin{array}{r} A\ 1\ 2\ B \\ -\ 3\ C\ D\ 4 \\ \hline 5\ 6\ 7\ 8 \end{array}$$

6 The pattern in the table repeats every three elements.

The 50th, 51st, 52nd and 53rd elements are shown.

What was the 2nd element of this pattern?

□△○	○□△	△○□	□△○
50th	51st	52nd	53rd

7 What is the value of the following expression?

2025 – 225 – 252 – 522

8 In my class there is one birthday in each of the months June, July, August and September.

Oliver's birthday is earlier than Amelia's.

Charlotte's birthday is 2 months after Jack's, but not in September.

Who has a birthday in July?

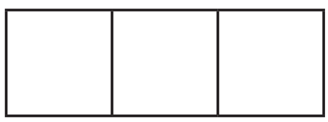
9 What is the value of the following expression?

(7 × 8) – (6 × 8) + (5 × 6) – (4 × 6) + (3 × 4) – (2 × 4)

10 The figure shows a shape that is made of 3 identical squares.

The perimeter of the shape is 80 centimetres.

Find the area of the shape, in square centimetres.



Peter’s Sandwiches

Problem:

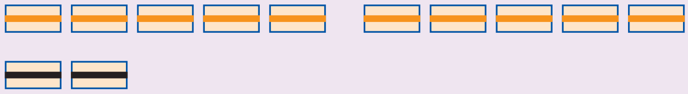
Peter made marmalade sandwiches and vegemite sandwiches.
He made 8 more marmalade sandwiches than vegemite sandwiches.
He made 30 sandwiches in total.
How many marmalade sandwiches did Peter make?

Strategy: Guess, Check and Refine

Let's guess that Peter made 10 marmalade sandwiches.

He's made **8** more marmalade than vegemite, so he's also made 10 – 8 = 2 vegemite sandwiches.

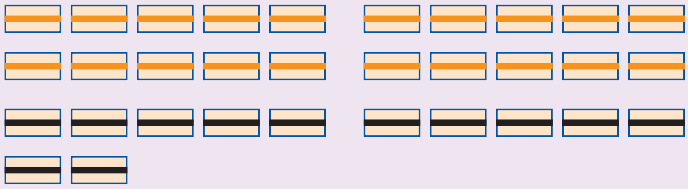
All together, Peter would have 10 + 2 = 12 sandwiches.



If Peter made 20 marmalade sandwiches, then he's also made 20 – 8 = 12 vegemite sandwiches.

All together, Peter would have 20 + 12 = 32 sandwiches.

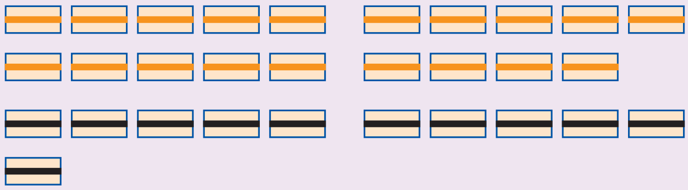
That's too many, but it's pretty close.



If Peter made 19 marmalade sandwiches, then he's also made 19 – 8 = 11 vegemite sandwiches.

All together, Peter would have 19 + 11 = 30 sandwiches.

That matches the question.



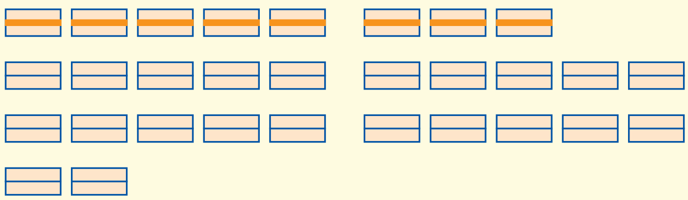
Peter made **19** marmalade sandwiches.

Strategy: Draw a Diagram

Peter made 8 more marmalade sandwiches than vegemite sandwiches.

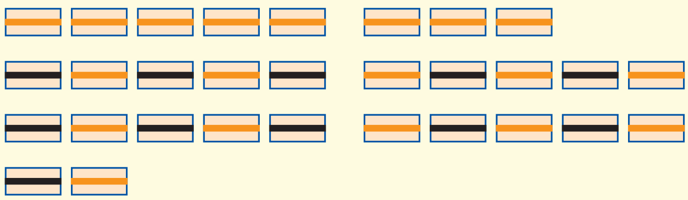
Let's suppose he made those 8 marmalade sandwiches first.

He then set out the bread, ready to make the other 30 – 8 = 22 sandwiches.



If Peter now makes equal numbers of vegemite and marmalade sandwiches, he will continue to have 8 more marmalade sandwiches than vegemite sandwiches.

He might do this by making the sandwiches in pairs.



In his sandwich pairs, Peter has made 22 ÷ 2 = 11 marmalade and 22 ÷ 2 = 11 vegemite sandwiches.

In total, Peter made **8 + 11 = 19** marmalade sandwiches.

Problem: Martin, Nick, Olive and Penny are waiting in a line.

Nick is between Olive and Martin.

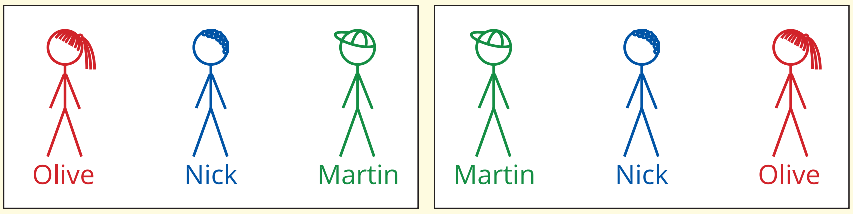
Martin is next to Penny.

If Penny is third in the line, who is first in line?

Strategy: Draw a Diagram

Nick is between Olive and Martin.

There are **2** ways that this could happen.

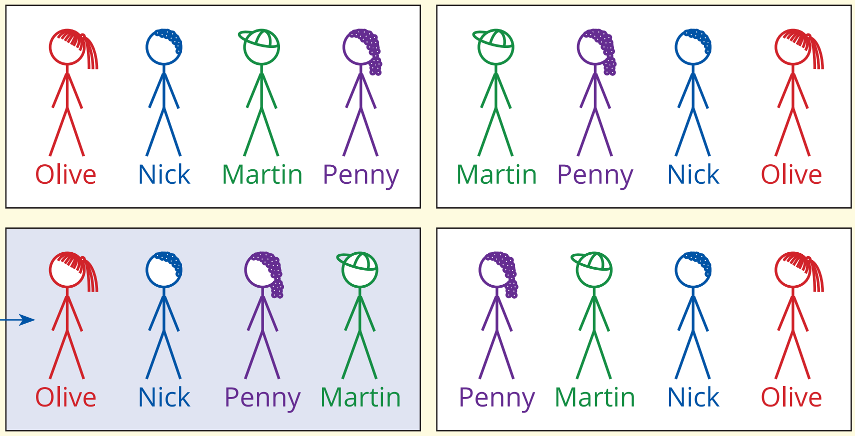


Martin is next to Penny.

Martin could be either in front of, or behind Penny.

Since there are 2 possible positions for Martin, there are now **2 × 2 = 4** possible arrangements.

Out of these **4** arrangements, there is only one where Penny is third in line.

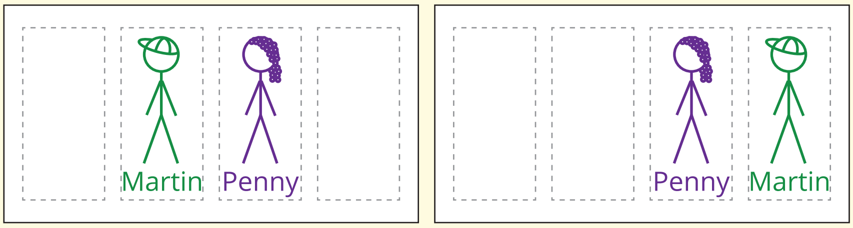


When Penny is third in line, the person who is first in line is **Olive**.

Strategy: Draw a Diagram (Alternative Approach)

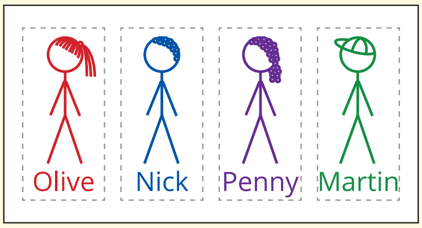
Penny is third in line, and Martin is next to Penny.

This can occur in two ways.



Nick is between Olive and Martin.

This only makes sense in one of the above scenarios.



We can see that **Olive** is first in line.

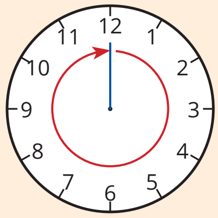
125 Hours Later

Problem: Suppose the time is now 5 o’clock on a 12-hour clock.
What time will this clock show 125 hours from now?

Strategy: Find a Pattern

125 hours is not a convenient amount to add. What is an easy number of hours to add?

How about 12 hours? This might be more convenient, because it takes 12 hours for the time on a 12-hour clock to complete a full cycle.



Method 1: Count a 12-hour clock cycle from 12 o'clock to 12 o'clock.

We can build a table to keep track of how many hours have passed.

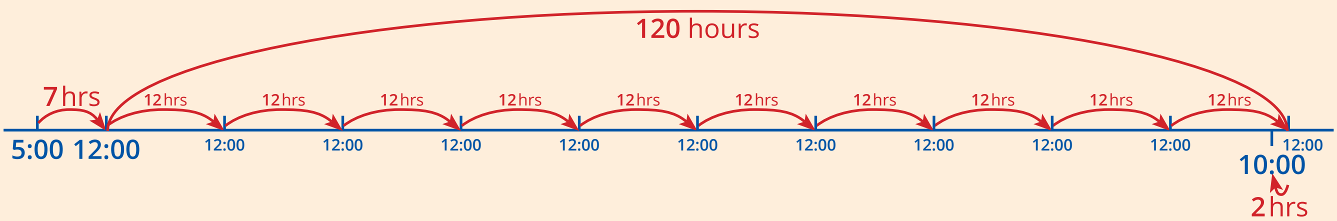
It may help to think of a 12-hour cycle from 12 o'clock to 12 o'clock.
7 hours after 5 o'clock, it will be 12 o'clock.

Every 12 hours after that, it will be 12 o'clock again.
To get close to 125 hours in total, let's add 10 × 12 = 120 hours.

We've gone over by 127 – 125 = 2 hours.
2 hours before 12 o'clock, the time would have been 10 o'clock.

Time Now	Hours to add	Total hours	New time
5:00	+7	7	12:00
12:00	+120	127	12:00
12:00	–2	125	10:00

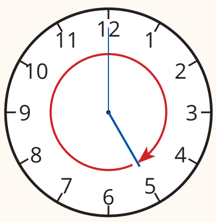
This method can also be represented as a time line.



We can see that, 125 hours after 5 o'clock, the clock is going to show **10 o'clock**.

Method 2: Count a 12-hour clock cycle from 5 o'clock to 5 o'clock.

If it's **5 o'clock** now, in 12 hours' time the clock will show **5 o'clock** again.



We want to find the time 125 hours from now.

125 ÷ 12 = 10 r.5.

In 10 × 12 = 120 hours' time the clock will likewise show **5 o'clock**.

5 hours after that, 120 + 5 = 125 hours will have passed.

125 hours after **5 o'clock**, the clock will show **10 o'clock**.

Equilateral Triangle Pattern

Problem: I am making a pattern out of equilateral triangles.

Each triangle has a perimeter of 3 cm.

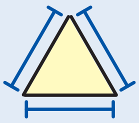
The diagrams show what the pattern looks like with 1, 2, 3, and 4 triangles.

Find the perimeter, in centimetres, for a pattern made up of 10 triangles.

Strategy: Build a Table

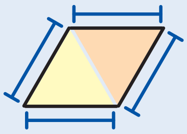
One equilateral triangle has a perimeter of **3** cm.

Since the triangle has three equal sides, each side length is **3 ÷ 3 = 1** cm.



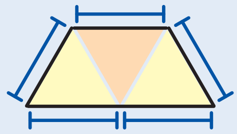
When the pattern is made up of 2 triangles, the perimeter is made up of **4** side lengths.

The perimeter of the 2-triangle pattern is **4** cm.



When the pattern is made up of 3 triangles, the perimeter is made up of **5** side lengths.

The perimeter of the 3-triangle pattern is **5** cm.



We can build a table to show how the perimeter grows.

+1 +1

Triangles	1	2	3
Perimeter (cm)	3	4	5

+1 +1

Adding one triangle increases the perimeter by **1** cm.

We can continue the table to find the perimeter when there are 10 triangles.

+1 +1 +7

Triangles	1	2	3	...	10
Perimeter (cm)	3	4	5	...	12

+1 +1 +7

When there are **10** triangles in the pattern, the perimeter is **12** cm.

Alternatively, the number of centimetres in the perimeter is **2** more than than the number of triangles.

+1 +1

Triangles	1	2	3		...	10
Perimeter (cm)	3	4	5		...	12

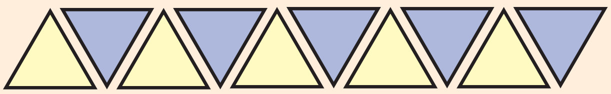
+1 +1 (+2, +2)

When there are **10** triangles in the pattern, the perimeter is **10 + 2 = 12** cm.

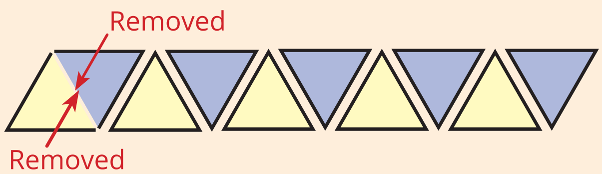
Strategy: Find a Pattern

Each triangle has a perimeter of **3** cm.

With 10 triangles, the total perimeter if the triangles are separate would be 10 × **3** = **30** cm.

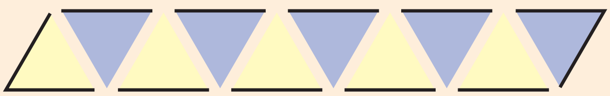


When we join two triangles together, the total perimeter is reduced by **2** side lengths, or **2** cm.

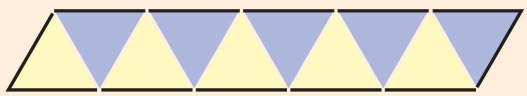


A pattern with 10 triangles has **9** joins.

Each join reduces the total perimeter by **2** cm.



When there are **10** triangles in the pattern, the perimeter is **30 – (9 × 2) = 12** cm.



Finding ABCD

Problem:

In the subtraction shown, letters are used to represent digits.
Different letters do NOT necessarily represent different digits.
What is the four-digit number represented by ***ABCD***?

$$\begin{array}{r} A\ 1\ 2\ B \\ -\ 3\ C\ D\ 4 \\ \hline 5\ 6\ 7\ 8 \end{array}$$

Strategy: Work Backwards

Working from the ones column, *B* – 4 must be a number that ends in 8.

If *B* – 4 = 8
then *B* = 8 + 4
= 12.

Since *B* is a one-digit number, *B* must equal 2, with trading from the tens column.

$$\begin{array}{r} A\ 1\ 2\ B \\ -\ 3\ C\ D\ 4 \\ \hline 5\ 6\ 7\ 8 \end{array} \qquad \begin{array}{r} A\ 1\ \overset{1}{\not 2}\ \overset{1}{B} \\ -\ 3\ C\ D\ 4 \\ \hline 5\ 6\ 7\ 8 \end{array}$$

In the tens column, 1 – *D* must be a number that ends in 7.

While 1 – *D* = 7 does not make sense for a subtraction algorithm, we can have 11 – *D* = 7.

Then, *D* = 11 – 7, with trading from the hundreds column.
D = 4.

$$\begin{array}{r} A\ 1\ \overset{1}{\not 2}\ \overset{1}{2} \\ -\ 3\ C\ D\ 4 \\ \hline 5\ 6\ 7\ 8 \end{array} \qquad \begin{array}{r} A\ \overset{0}{\not 1}\ \overset{11}{\not 2}\ \overset{1}{2} \\ -\ 3\ C\ D\ 4 \\ \hline 5\ 6\ 7\ 8 \end{array}$$

In the hundreds column, 0 – *C* must end in 6.

While 0 – *C* = 6 does not make sense for a subtraction algorithm, we can have 10 – *C* = 6, with trading from the thousands column.

Then, *C* = 10 – 6
= 4.

$$\begin{array}{r} A\ \overset{0}{\not 1}\ \overset{11}{\not 2}\ \overset{1}{2} \\ -\ 3\ C\ 4\ 4 \\ \hline 5\ 6\ 7\ 8 \end{array} \qquad \begin{array}{r} \overset{A-1}{\not A}\ \overset{10}{\not 1}\ \overset{11}{\not 2}\ \overset{1}{2} \\ -\ 3\ C\ 4\ 4 \\ \hline 5\ 6\ 7\ 8 \end{array}$$

In the thousands column, (*A* – 1) – 3 must end in 5.

If (*A* – 1) – 3 = 5
then *A* – 1 = 5 + 3
A = 5 + 3 + 1
= 9.

$$\begin{array}{r} \overset{A-1}{\not A}\ \overset{0}{\not 1}\ \overset{11}{\not 2}\ \overset{1}{2} \\ -\ 3\ 4\ 4\ 4 \\ \hline 5\ 6\ 7\ 8 \end{array} \qquad \begin{array}{r} \overset{9-1}{\not A}\ \overset{10}{\not 1}\ \overset{11}{\not 2}\ \overset{1}{2} \\ -\ 3\ 4\ 4\ 4 \\ \hline 5\ 6\ 7\ 8 \end{array}$$

We have: *A* = 9
B = 2
C = 4
D = 4

Let's check:

$$\begin{array}{r} A\ 1\ 2\ B \\ -\ 3\ C\ D\ 4 \\ \hline 5\ 6\ 7\ 8 \end{array} \qquad \begin{array}{r} 9\ 1\ 2\ 2 \\ -\ 3\ 4\ 4\ 4 \\ \hline 5\ 6\ 7\ 8 \end{array} \qquad \begin{array}{r} \overset{8}{\not 9}\ \overset{10}{\not 1}\ \overset{11}{\not 2}\ \overset{1}{2} \\ -\ 3\ 4\ 4\ 4 \\ \hline 5\ 6\ 7\ 8 \end{array}$$

The four-digit number represented by *ABCD* is **9244**.

Strategy: Work Backwards (Alternative Approach)

If *A*12*B* – 3*CD*4 = 5678, then 3*CD*4 + 5678 = *A*12*B*.

$$\begin{array}{r} 3\ C\ D\ 4 \\ +\ 5\ 6\ 7\ 8 \\ \hline A\ 1\ 2\ B \end{array}$$

In the ones column,
4 + 8 = 12.

We can see that *B* = 2, and the remaining 10 is added to the tens column.

$$\begin{array}{r} 3\ C\ \overset{1}{D}\ 4 \\ +\ 5\ 6\ 7\ 8 \\ \hline A\ 1\ 2\ 2 \end{array}$$

In the tens column, 1 + *D* + 7 = 2 won't work, but we can have

1 + *D* + 7 = 12
D = 4.

$$\begin{array}{r} 3\ \overset{1}{C}\ \overset{1}{4}\ 4 \\ +\ 5\ 6\ 7\ 8 \\ \hline A\ 1\ 2\ 2 \end{array}$$

In the hundreds column, 1 + *C* + 6 = 1 won't work, so

1 + *C* + 6 = 11
C = 4.

$$\begin{array}{r} \overset{1}{3}\ \overset{1}{4}\ \overset{1}{4}\ 4 \\ +\ 5\ 6\ 7\ 8 \\ \hline A\ 1\ 2\ 2 \end{array}$$

In the thousands column, 1 + 3 + 5 = *A* and so *A* = 9.

The four-digit number represented by *ABCD* is **9244**.

$$\begin{array}{r} \overset{1}{3}\ \overset{1}{4}\ \overset{1}{4}\ 4 \\ +\ 5\ 6\ 7\ 8 \\ \hline 9\ 1\ 2\ 2 \end{array}$$

Problem: The pattern in the table repeats every three elements.

The 50th, 51st, 52nd and 53rd elements are shown.

What was the 2nd element of this pattern?

□△○	○□△	△○□	□△○
50th	51st	52nd	53rd

Strategy: Make an Organised List

Since the pattern repeats every **3** elements, the **2**nd element is the same as the **2 + 3 = 5**th element in the pattern.

We can list all of the elements that are the same as the **2**nd element, until we reach an element that we know.

2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47, 50

+3 +3 +3 +3 +3 +3 +3 +3 +3 +3 +3 +3 +3 +3 +3 +3

The **2**nd element in the pattern is the same as the **50**th, which is □△○.

Strategy: Build a Table

When going from one element to the next in the pattern, the last symbol is moved to the front.

There are **3** shapes in each element.

□△○	○□△	△○□
50th	51st	52nd
□△○	○□△	△○□
53rd	54th	55th

We can construct a table to visualise the pattern.

Since we know elements **50 - 55**, and we want to find element **2**, we will need to count down from **55**.

55	52	49	46	43	40	37	34	31	28	25	22	19	16	13	10	7	4
54	51	48	45	42	39	36	33	30	27	24	21	18	15	12	9	6	3
53	50	47	44	41	38	35	32	29	26	23	20	17	14	11	8	5	2

2 is in the same row as **50**, so the **2nd** element will have the same pattern as the **50th** element.

The **2**nd element of the of this pattern was □△○.

Strategy: Find a Pattern

Since the pattern repeats for every **3**rd element, all elements that are a multiple of **3** will be the same.

□△○	○□△	△○□
53	54	55
56	57	58
59	60	

We know that **60 = 20 × 3**.

Since **60** is a multiple of **3**, and **3** is also a multiple of 3, we know that element **3** will look like element **60**.

Element 3 is ○□△.

Therefore, element 2 is □△○.

□△○	○□△	△○□
2	3	4
:	:	:
59	60	

2025 – 225 – 252 – 522

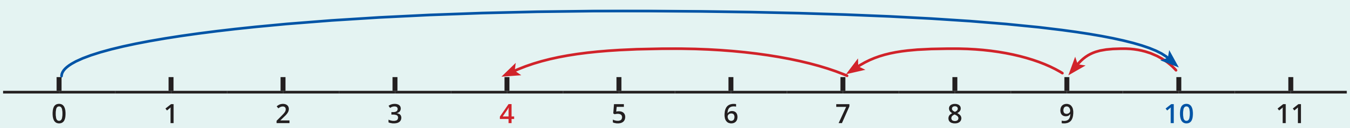
Problem: What is the value of the following expression?

2025 – 225 – 252 – 522

Strategy: Solve a Simpler Related Problem

To work out the value of 2025 – 225 – 252 – 522, we can begin by considering what it means to perform three subtractions in a row.

If the question had been, for example, 10 – 1 – 2 – 3, we can visualise it as shown.



The result is the same as 10 – (1 + 2 + 3).

Since

2025 – 225 – 252 – 522

= 2025 – (225 + 252 + 522):

	2 2 5
	2 5 2
+	5 2 2
	9 9 9

225 + 252 + 522 = 999.

The value we want is given by the expression 2025 – 999.

It is relatively simple to work out that 2025 – 1000 = 1025.

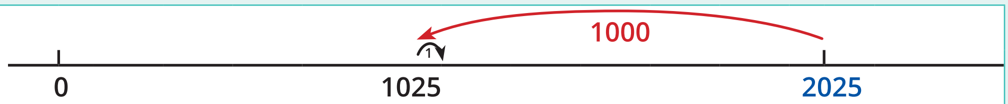
If we subtracted 999 instead of 1000, the result would be reduced by a smaller amount.

Therefore:

2025 – 225 – 252 – 522

= 2025 – 999

= 2025 – 1000 + 1.



The value of 2025 – 1000 + 1 is 1025 + 1 = **1026**.

Strategy: Use a Written Algorithm

To find the value of 2025 – 225 – 252 – 522, we can perform the subtractions one a time.

2025 – 225 – 252 – 522 = **1026**.

	¹~~2~~ ¹0 2 5
–	2 2 5
	1 8 0 0

	1 ⁷~~8~~ ⁹~~0~~ ¹0
–	2 5 2
	1 5 4 8

	1 5 4 8
–	5 2 2
	1 0 2 6

June, July, August, September

Problem: In my class there is one birthday in each of the months June, July, August and September.

Oliver’s birthday is earlier than Amelia’s.

Charlotte’s birthday is 2 months after Jack’s, but not in September.

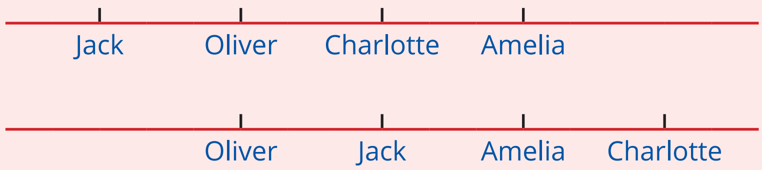
Who has a birthday in July?

Strategy: Eliminate All But One Possibility

Oliver's birthday is earlier than Amelia's.

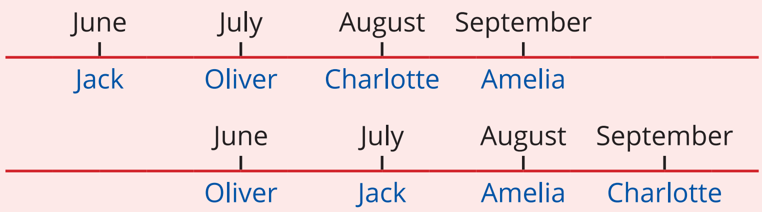
Charlotte's birthday is two months after Jack's.

Since the students' birthdays are in consecutive months, either Oliver's birthday or Amelia's birthday must be between Jack's birthday and Charlotte's.



We can now assign months to each of the two possible scenarios.

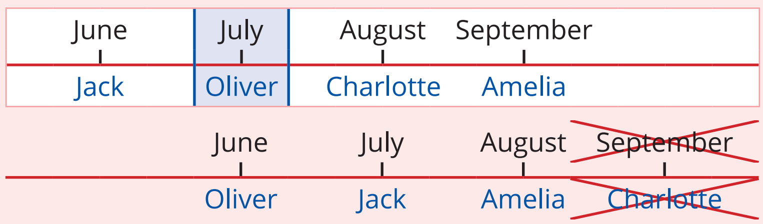
The students' birthdays are in June, July, August and September.



Charlotte's birthday is not in September.

This eliminates one possible scenario.

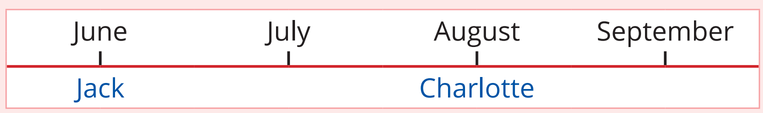
The student with a birthday in July must be **Oliver**.



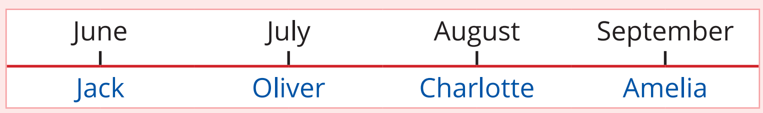
Strategy: Eliminate All But One Possibility (Alternative Approach)

Charlotte's birthday is two months after Jack's, but not in September.

For this to be true, Jack and Charlotte's birthdays must be in June and August.



Oliver's birthday is earlier than Amelia's.



The student with a birthday in July is **Oliver**.

Simplifying Using Factors

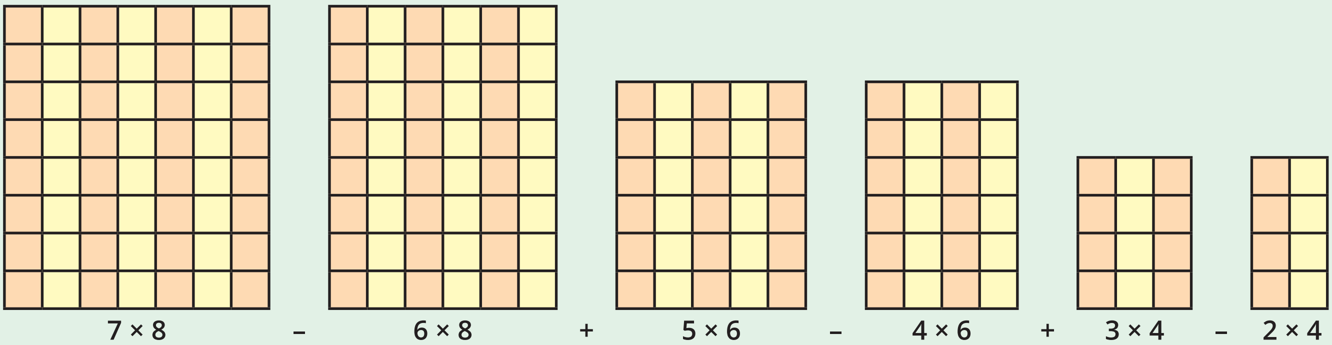
Problem:

What is the value of the following expression?

$(7 \times 8) - (6 \times 8) + (5 \times 6) - (4 \times 6) + (3 \times 4) - (2 \times 4)$

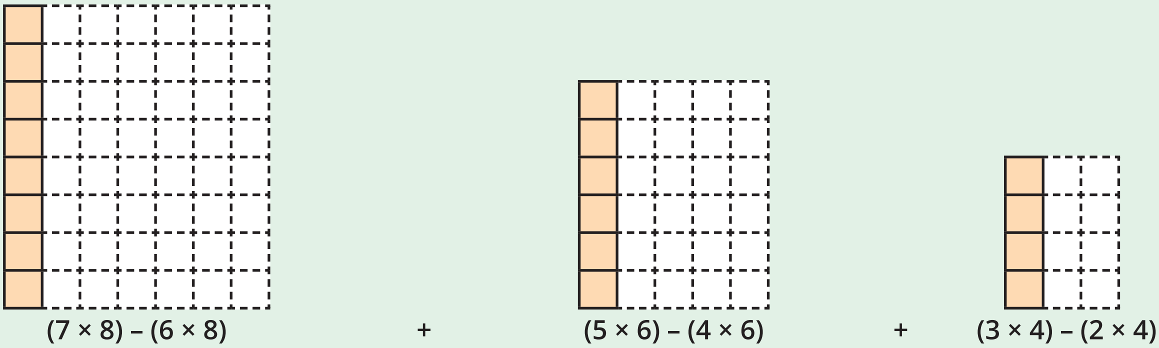
Strategy: Convert to a More Convenient Form

To make this problem easier to think about, we can represent each multiplication as a diagram.



Each pair of products shares a factor, which is represented in the diagram as the height of the rectangle.

We can use this to work out the difference between two consecutive terms.



The value of the expression is **8 + 6 + 4 = 18**.

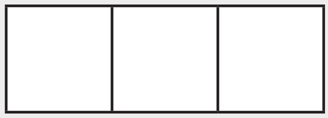
Strategy: Reason Arithmetically

We can solve each multiplication individually before subtracting and then adding them.

7 × 8	–	6 × 8	+	5 × 6	–	4 × 6	+	3 × 4	–	2 × 4
56	–	48	+	30	–	24	+	12	–	8
8			+	6			+	4		

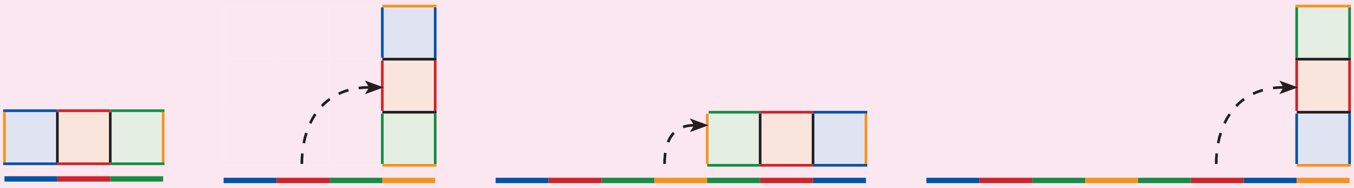
The value of the expression is **8 + 6 + 4 = 18**.

Problem: The figure shows a shape that is made of 3 identical squares.
The perimeter of the shape is 80 centimetres.
Find the area of the shape, in square centimetres.



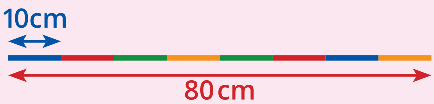
Strategy: Divide a Complex Shape

Let's roll out the perimeter of the shape so that we can see it as a straight line.



The perimeter is made up of 8 identical line segments, each the length of the side of one square.

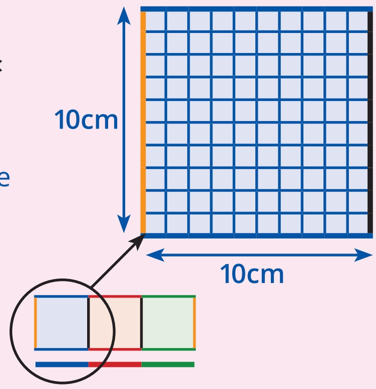
We know that the perimeter of the shape is 80 centimetres, so the length of one line segment must be 80 ÷ 8 = 10 cm.



Each square has a side length of 10 cm.

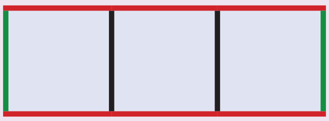
One square will have an area of 10 cm × 10 cm = 100 cm^2.

Since the shape comprises three squares, the area of the shape would be 3 × 100 cm^2 = **300** cm^2.



Strategy: Guess, Check and Refine

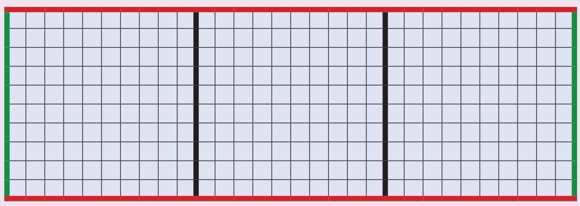
We could guess a height for the shape, and see if it matches the conditions for the problem.



Height (cm)	Width (cm)	Perimeter (cm)
5	3 × 5 = 15	15 + 5 + 15 + 5 = 40
6	3 × 6 = 18	18 + 6 + 18 + 6 = 48
7	3 × 7 = 21	21 + 7 + 21 + 7 = 56
8	3 × 8 = 24	24 + 8 + 24 + 8 = 64
9	3 × 9 = 27	27 + 9 + 27 + 9 = 72
10	3 × 10 = 30	30 + 10 + 30 + 10 = 80

When the height is 10 cm, the perimeter of the shape is **80** cm. This matches the problem.

With a height of 10 cm, the width will be 30 cm.



Therefore the area of the shape is 30 cm × 10 cm = **300** cm^2.

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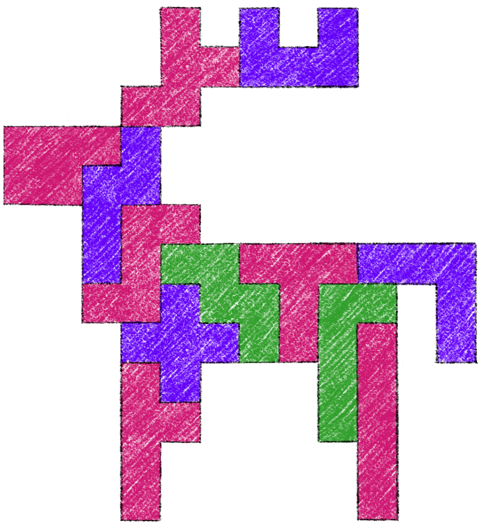
Guess, Check and Refine



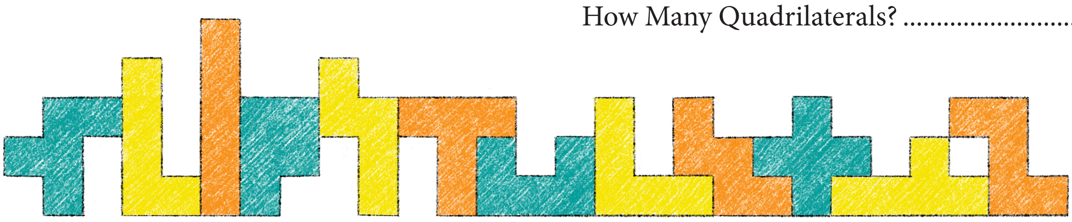
Draw a Diagram

Find a Pattern

Work Backwards

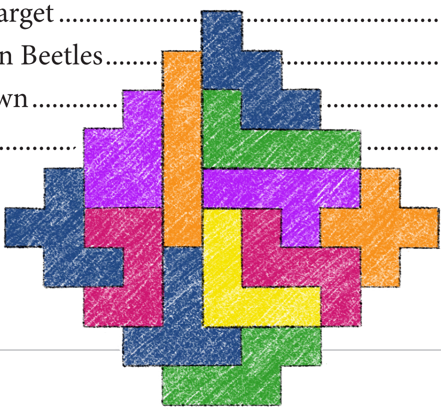
Build a Table

Make an Organised List



Solve a Simpler Related Problem

Convert to a More Convenient Form



Eliminate All But One Possibility

Divide a Complex Shape

Further Methods