Important

The information contained in this file is ONLY for the use of registered participants of the 2024 Australasian Problem Solving Mathematical Olympiads.

APSMO
2024 OLYMPIADS

Organisation and Procedures

For full details, see the Members' Area

To ensure the integrity of the competition, the Olympiads must be administered under examination conditions.

DO	DO NOT
• Supervise students at all times • Seat students apart • Maintain silence • Provide blank working paper • Give time warnings when 3 minutes remain, and again when 1 minute remains • Collect, mark and retain the papers	• Print the Olympiad papers prior to the Olympiad Date • Read the questions aloud to the students • Interpret the questions for students • Permit any discussion or movement around the room • Permit the use of calculators or other electronic devices

- Olympiad papers are scored by the PICO using the *Solutions and Answers* sheet provided.
- Results should be submitted in the Members' Area within 7 days of the Olympiad.
- Original student answer sheets should be retained by the PICO until the end of the year.
- *Solutions and Answers sheets* are not to be handed out to students. They are a teaching resource for use in class ***after*** completion of the Olympiad paper.

Timing of the Olympiad

- The *Total Time Allowed* for the Olympiad is **30 minutes**.
- The time for each individual question is a guide for the students.

Absent Student Policy

A student who is legitimately absent on the Olympiad date, may sit the Olympiad under examination conditions on their first day back at school (if that date is within 2 weeks of the original Olympiad date). If these conditions cannot be met, the student must be marked as absent on the submitted results.

The Absent Student Policy is available in the **Contest Administration** section of the Members' Area.

APSMO

2024 : DIVISION S

WEDNESDAY 31 JULY 2024

OLYMPIAD 3

Total Time Allowed: ***30 Minutes***

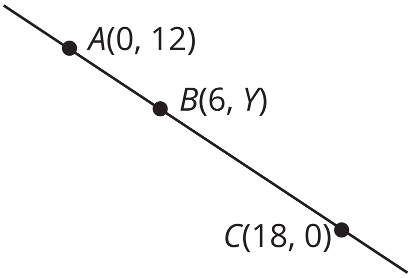
Write your answers in the boxes on the back.

Keep your answers hidden by folding backwards on this line.

3A. The points A, B, and C lie on a straight line.

Point A has coordinates (0, 12), point B has coordinates (6, Y), and point C has coordinates (18, 0), as shown.

Find the value of Y.



3B. The product $3^{2024} \times 7^{2023}$ is expressed as a whole number.

What is the digit in the ones place?

3C. In a recent survey, only respondents who indicated that they loved pizza, loved salads, or loved both, were considered.

Two-thirds of pizza lovers are salad lovers.

One-half of salad lovers are pizza lovers.

What percentage of those surveyed loved both pizza and salad?

3D. x and y are both positive integers.

$x + xy + y = 17$.

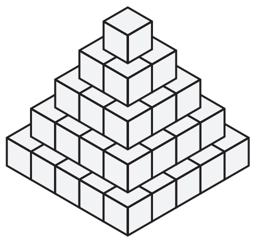
Find the smallest possible value of $x + y$.

3E. A 5-tier square-base tower is constructed using 5 centred layers of identical cubes.

The number of cubes in each tier is a perfect square.

The entire tower is then fully immersed in a bucket of blue paint.

How many of the individual cubes have exactly 25% of their surface area coloured blue?



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OLYMPIAD

3

3A.

Student Name:

Fold here. Keep your answers hidden.

3B.

3C.

3D.

3E.

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OLYMPIAD 3

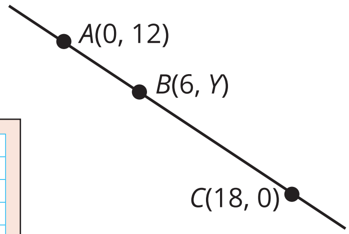
Solutions and Answers

(Items in parentheses are not required)

For teacher use only. Not for Distribution.

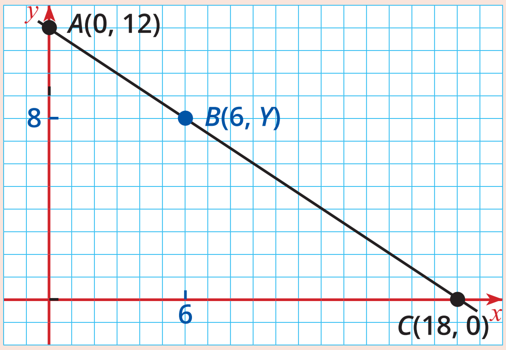
3A: 8	**3B:** 3	**3C:** 40 (%)	**3D:** 7	**3E:** 12

3A. The question is: Find the value of *Y*.



METHOD 1: *Draw a diagram.*

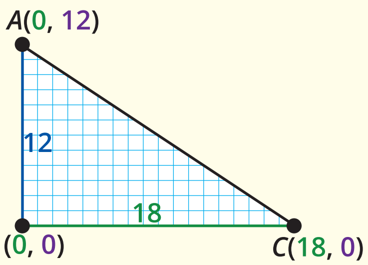
By drawing the line accurately on grid paper, by inspection, we can see that the value of ***Y*** is **8**.



METHOD 2: *Consider the slope of the line.*

Line *AC* forms the hypotenuse of a triangle that is:

- 18 units wide, and
- 12 units high.

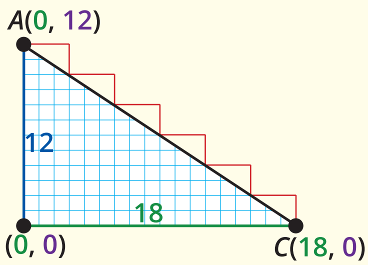


To get from ***A*** to ***C***, we would:

- Move 18 units to the right, and
- Move 12 units down.

Since both 18 and 12 are divisible by 6, we can divide line *AC* into 6 sections, so that, for each section, we would:

- Move 18 ÷ 6 = 3 units to the right, and
- Move 12 ÷ 6 = 2 units down.

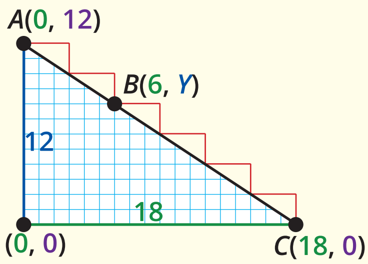


To get from ***A*** to ***B***, we would need to move 2 × 3 = 6 units to the right.

Every time we move 3 units to the right, we move 2 units down, so in total we would move 2 × 2 = 4 units down.

Therefore the y-coordinate of ***B*** is 12 – 4 = 8.

Since the coordinates of ***B*** are (6, *Y*), the value of ***Y*** is **8**.



Follow-Up*: Given the original question, assume that point D is on the same line. Find the coordinates of D, such that the length of DC is the same as the length of AD.* [(9, 6)]

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OLYMPIAD 3

3B. The question is: For $3^{2024} \times 7^{2023}$, what is the digit in the ones place?

METHOD 1: *Convert to a more convenient form, and find a pattern.*

We want to find the digit in the ones place for

$3^{2024} \times 7^{2023} = \underbrace{3 \times 3 \times 3 \ldots\ldots \times 3}_{2024\text{ times}} \times \underbrace{7 \times 7 \times 7 \ldots\ldots \times 7}_{2023\text{ times}}$.

Since multiplication is commutative and associative, we can reframe the problem as

$\underbrace{(3 \times 7) \times (3 \times 7) \times (3 \times 7) \times \ldots \times (3 \times 7)}_{2023\text{ times}} \times 3$

$= \underbrace{21 \times 21 \times 21 \times \ldots \times 21}_{2023\text{ times}} \times 3.$

Using a table to investigate $\underbrace{21 \times 21 \times 21 \times \ldots \times 21}_{2023\text{ times}}$:

21^2	21^3	21^4
441	9261	194481

```
  □ ... □ □ 1
          2 1 ×
  -----------
  □ ... □ □ 1
□ □ ... □ □ 0
  -----------
□ □ ... □ □ 1
```

We notice that, for every power of **21**, the digit in the ones place is **1**.

By inspecting the multiplication algorithm, we can be sure that this pattern will continue.

21^{2023} would also have **1** in the ones place.

To find the digit in the ones place for $21^{2023} \times 3$, we can once again consider what will occur when we execute the multiplication algorithm.

Irrespective of the digits in the other place values, the digit in the ones place of the product will be **3**.

```
  □ □ ... □ □ □ □ 1
                  3 ×
  -----------------
□ □ □ ... □ □ □ □ 3
```

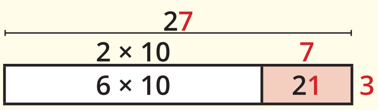
METHOD 2: *Find a pattern.*

For powers of 3, the ones place digit cycles through the pattern 3, 9, 7, 1, 3, 9, 7,1, ... with the ones digit repeating for every **4**th value.

	Value
3^1	3
3^2	9
3^3	27
3^4	81
3^5	243
3^6	729
3^7	2187
3^8	6561

This makes sense when we consider that the ones digit of a product is solely determined by the ones digits of the two multiplicands.

Digits in higher place values represent multiples of **10**, and so have no effect on the ones digit.



We note that the ones digit for 3^4 is **1**, and the ones digit for 3^8 is also **1**.

2024 is a multiple of **4**, so from the pattern we can see that the ones digit for 3^{2024} is also **1**.

For powers of 7, the ones place digit cycles through the pattern 7, 9, 3, 1, 7, 9, ... with the ones digit repeating for every **4**th value.

	Value
7^1	7
7^2	49
7^3	343
7^4	2401
7^5	16807
7^6	...9

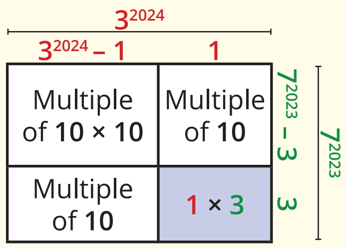
Since **2024** is a multiple of **4**, from the pattern we can see that the ones digit for 7^{2024} is **1**.

The ones digit for 7^{2023} would be 3, since 3 precedes 1 in the pattern.

The ones digit for 3^{2024} is 1, and for 7^{2023} is 3, so $(3^{2024} - 1)$ and $(7^{2023} - 3)$ are both multiples of **10**.

From the area diagram, we can see that ones digit in $3^{2024} \times 7^{2023}$ would be the ones digit in the product 1 × 3.

The digit in the ones place is **3**.



Follow-Up: *Consider all products of the form $2^a \times 3^b$. How many different digits could be the ones digit of such a product?* [4]

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OLYMPIAD 3

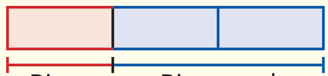
3C. The question is: What percentage of those surveyed loved both pizza and salad?

METHOD 1: *Draw a diagram.*

We can use a bar to represent the number of pizza lovers.

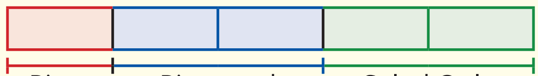
Two-thirds of pizza lovers are salad lovers.

This means that the other one-third of pizza lovers only love pizza - they do not love salads.



One-half of salad lovers are pizza lovers.

The respondents we have already marked as saying they love both pizza and salad, must be one-half of all salad lovers.



The same number of respondents to the survey must have said that they love salad only.

From the diagram, we can see that two-fifths, or **40**%, of the respondents, loved both pizza and salad.

METHOD 2: *Reason algebraically.*

Let x represent the number of respondents who loved both pizza and salad.

Let y represent the number of respondents who loved pizza only.

The total number of pizza lovers would therefore be $x + y$.

Two-thirds of pizza lovers are salad lovers, so x is two-thirds of $(x + y)$.

	x	=	$\frac{2}{3}(x + y)$
Multiply both sides by 3:	$3x$	=	$2(x + y)$
		=	$2x + 2y$
Subtract $2x$ from both sides:	x	=	$2y$
Finding y in terms of x:	y	=	$\frac{1}{2}x$

Let z represent the number of respondents who loved salad only.

The total number of salad lovers would therefore be $x + z$.

One-half of salad lovers are pizza lovers, so x is one-half of $(x + z)$.

	x	=	$\frac{1}{2}(x + z)$
Multiply both sides by 2:	$2x$	=	$x + z$
Subtract x from both sides:	x	=	z
Finding z in terms of x:	z	=	x

$x + y + z$ represents **100%** of respondents to the survey.

Substituting $y = \frac{1}{2}x$ and $z = x$, we can determine that x comprises **40%** of respondents.

	$x + y + z$	=	**100%** of respondents
Substituting $y = \frac{1}{2}x$, $z = x$:	$x + \frac{1}{2}x + x$	=	**100%** of respondents
	$\frac{5}{2}x$	=	**100%** of respondents
Multiply both sides by $\frac{2}{5}$:	x	=	**40%** of respondents

Follow-Up: *If we include respondents to the survey who did not like either pizza or salad, we find that only 25% of respondents liked both pizza and salad. What fraction did not like either pizza or salad?* [3/8]

3D. The question is: Find the smallest possible value of $x + y$, if $x + xy + y = 17$.

METHOD 1: *Make an organised list.*

Since the aim is to find the smallest possible value of $x + y$, we will list all possible values of $x + y$, starting with **2**.

We will consider positive integers x and y by listing the possible values of x starting with **1**, and incrementing by **1**, for values of $x \leq y$.

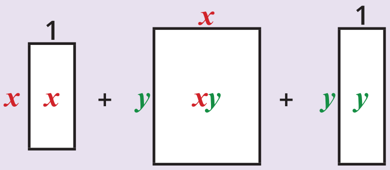
$x + y$	2	3	4	4	5	5	6	6	6	7	7	
x	1	1	1	2	1	2	1	2	3	1	2	
y	1	2	3	2	4	3	5	4	3	6	5	
xy	1	2	3	4	4	6	5	8	9	6	10	
$x + xy + y$	3	5	7	8	9	11	11	14	15	13	17	

Listing in this way, the first values of x and y for which $x + xy + y = 17$ are **2** and **5**.

The smallest possible value of $x + y$ is **7**.

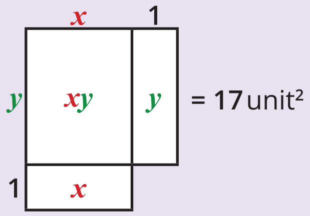
METHOD 2: *Draw a diagram.*

We can represent $x + xy + y$ using three separate area diagrams, as shown.

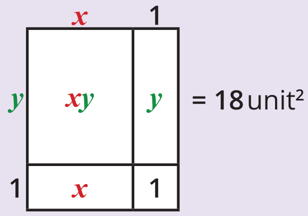


The sum of these areas is **17** unit².

The three diagrams can be combined into a single diagram with area **17** unit².



If we add an extra **1** unit² area to the diagram, the result is a rectangle with area **18** unit².



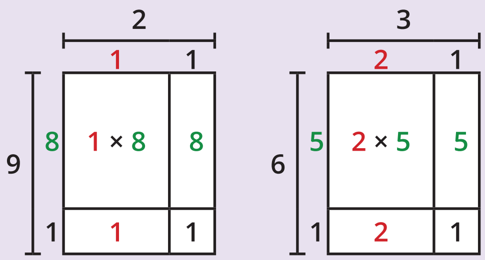
Possible dimensions for the rectangle are **2 × 9** and **3 × 6**.

Note that **1 × 18** is impossible, because x must be positive.

The possible values for x and y are **1** and **8**, or **2** and **5**.

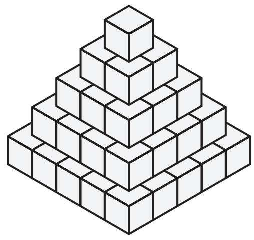
Of these two options, $x + y$ is smaller when x = **2** and y = **5**.

The smallest possible value of $x + y$ is **2 + 5 = 7**.



Follow-Up: *Suppose a and b are both positive integers. Find the greatest possible value of a + b, if a + ab + b = 19.* [9 + 1 = 10]

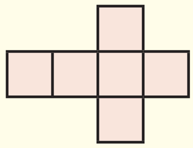
3E. The question is: How many of the individual cubes have exactly 25% of their surface area coloured blue?



METHOD: *Draw a diagram, and make an organised list.*

The surface area of a cube can be represented by unfolding the cube into its net.

It is equal to the combined areas of the **6** square faces.

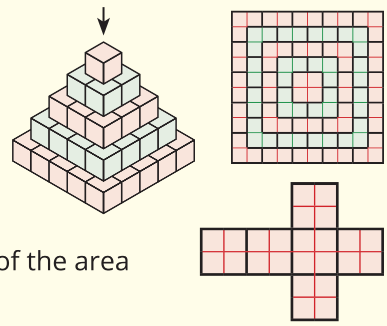


Since the cubes are centred on the layer below, we can see that the top faces of some cubes are fractionally exposed to the blue paint.

A top-down view of the tower shows that the top faces can have one-quarter or one-half of their surface obscured by the cube above.

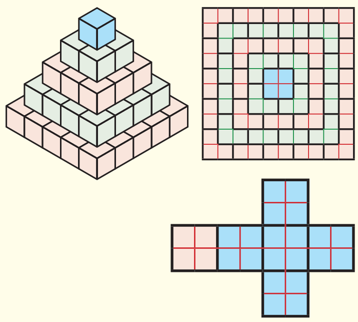
To work out the fraction of the surface area of a cube that is coloured blue, we will divide each face on the net into four small squares.

Each net has **6 × 4 = 24** small squares. We want to find cubes with **25%** of the area coloured blue; such cubes will have **25% × 24 = 6** small blue squares.



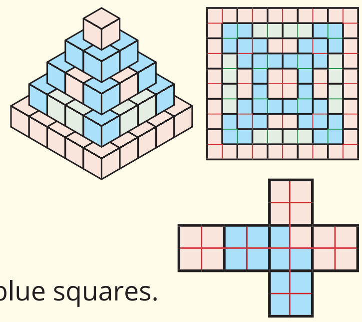
The topmost cube would have **5** faces painted blue.

This is equivalent to **5 × 4 = 20** small blue squares.



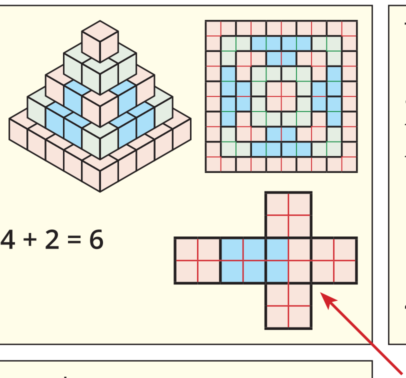
The corner cubes on every middle layer will have 2 whole faces, plus three-quarters of the top face, painted blue.

This is equivalent to **4 + 4 + 3 = 11** small blue squares.



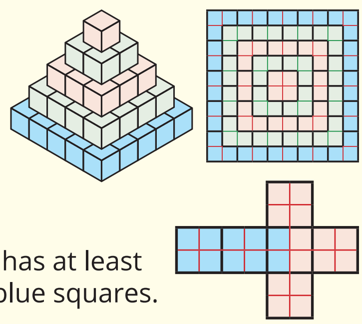
The side cubes on every middle layer will have one whole face, plus half of the top face, painted blue.

This is equivalent to **4 + 2 = 6** small blue squares.



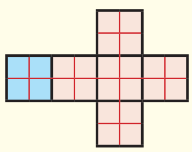
The visible cubes on the bottom layer all have the bottom face, one vertical face, and at least half of the top face, painted blue.

Each of these cubes has at least **4 + 4 + 2 = 10** small blue squares.



The remaining cubes on the bottom layer only have the bottom face painted blue.

Each of these cubes has only **4** small blue squares.



Of all of the cubes on the surface of the tower, the only ones with **6** small blue squares are on the sides of every middle layer.

There are **12** cubes with exactly **25**% of their surface painted blue.

Follow-Up: *How many cubes have more than 50% of their surface area coloured blue?* [5]