APSMO

2024 : DIVISION S

WEDNESDAY 4 SEPTEMBER 2024

MATHS OLYMPIAD

OLYMPIAD 4

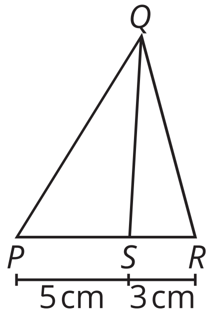
Total Time Allowed: ***30 Minutes***

Write your answers in the boxes on the back.

Keep your answers hidden by folding backwards on this line.

4A. N% of the prime numbers less than 40 contain the digit 7.
Find N.

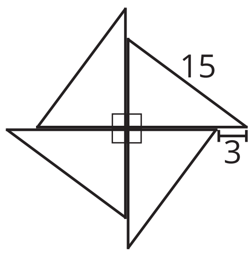
4B. In $\triangle PQR$, point S lies on side PR.
The length of PS is 5 cm.
The length of SR is 3 cm.
The area of $\triangle PQR$ is 36 square cm.
Find the area of $\triangle SQR$, in square cm.



4C. Express, as a decimal, the difference between $\sqrt{0.01}$ and $\frac{0.013}{0.01}$.

4D. A wooden cube is painted white and then cut into unit cubes.
Each unit cube measures 1 cm × 1 cm × 1 cm.
There are forty-eight unit cubes with exactly two faces painted white.
How many of the unit cubes have exactly one face painted white?

4E. The given star pattern is composed of four identical right-angled triangles.
The longest side of each right-angled triangle has length 15 cm.
The other two sides differ in length by 3 cm, as shown.
Find the total area of the star pattern, in square cm.



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OLYMPIAD

4

Student Name:

4A.

4B.

4C.

4D.

4E.

Fold here. Keep your answers hidden.

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OLYMPIAD 4

MATHS OLYMPIAD

Solutions and Answers

(Items in parentheses are not required)

For teacher use only. Not for Distribution.

4A: 25	**4B:** 13.5 (cm^2)	**4C:** 1.2	**4D:** 96	**4E:** 216 (cm^2)

4A. The question is: Find *N*, if *N*% of the prime numbers less than 40 contain the digit 7.

METHOD: *Make an organised list.*

A number is prime if it has exactly two factors, **1** and itself.

One way to find all of the prime numbers less than **40** is by eliminating multiples of numbers that have been identified to be prime (the Eratosthenes Sieve method).

1 has exactly one factor, and so it is not prime.

The first prime number is 2.

Any further multiples of 2 would not be prime, because they have 2 as a factor.

~~1~~	**2**	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~
31	~~32~~	33	~~34~~	35	~~36~~	37	~~38~~	39	

3 is the next whole number, and since it has not been eliminated, it must also be prime.

We can then eliminate any further multiples of 3.

Similarly, we can see that 5 is prime, and we can eliminate further multiples of 5.

~~1~~	**2**	**3**	~~4~~	**5**	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	

Using this method, or otherwise, we find that the prime numbers less than **40** are:

2, 3, 5, **7**, **11**, **13**, **17**, **19**, **23**, **29**, **31**, and **37**.

~~1~~	**2**	**3**	~~4~~	**5**	~~6~~	**7**	~~8~~	~~9~~	~~10~~
11	~~12~~	**13**	~~14~~	~~15~~	~~16~~	**17**	~~18~~	**19**	~~20~~
~~21~~	~~22~~	**23**	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	**29**	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	**37**	~~38~~	~~39~~	

There are **3** prime numbers less than **40** that contain the digit **7**:

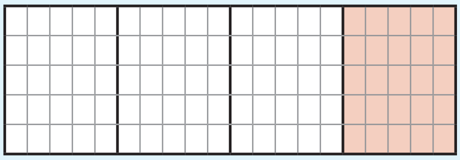
7, **17**, and **37**.

There are **12** prime numbers in total, that are less than **40**:

2, 3, 5, **7**, **11**, **13**, **17**, **19**, **23**, **29**, **31**, and **37**.

$\frac{3}{12}$, or $\frac{1}{4}$, of the prime numbers less than **40** contain the digit **7**.

2	3	5	7
11	13	19	17
23	29	31	37



Since $\frac{1}{4} = \frac{25}{100}$, we can see that **25**% of the prime numbers less than **40** contain the digit **7**.

The value of ***N*** is **25**.

Follow-Up: *K% of the prime numbers less than 100 contain the digit 2. Find K.* [12]

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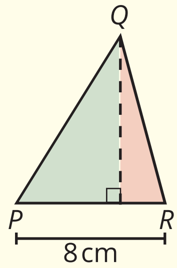
4B. The question is: Find the number of square units in the area of $\triangle SQR$.

METHOD 1: *Convert to a more convenient form.*

$\triangle PQR$ has an area of **36** cm^2.

Suppose we cut $\triangle PQR$ into two right-angled triangles through point Q, as shown.

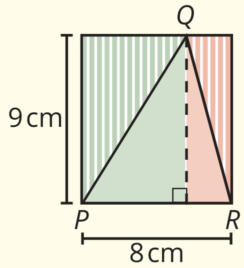


By duplicating both of these triangles, we can create a rectangle with base PR.

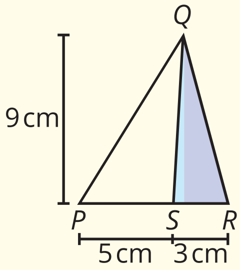
The area of this rectangle will be **2** × **36** cm^2 = **72** cm^2.

Therefore the height of the rectangle will be **72** cm^2 ÷ **8** cm = **9** cm.

This is also the height of $\triangle PQR$.



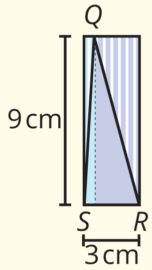
$\triangle SQR$ has the same height as $\triangle PQR$, so its height is also **9** cm.



A rectangle with height **9** cm, and with base SR, will have an area of **3** cm × **9** cm = **27** cm^2.

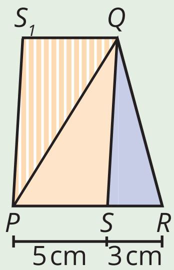
As seen above for $\triangle PQR$, a triangle with the same base and height will have half of the area of this rectangle.

The area of $\triangle SQR$ is **27** cm^2 ÷ **2** = **13.5** cm^2.



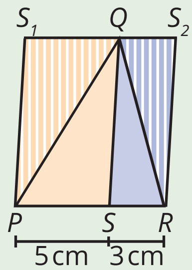
METHOD 2: *Construct a bounding parallelogram.*

We can begin by duplicating $\triangle PQS$ and rotating it to form parallelogram PS_1QS.

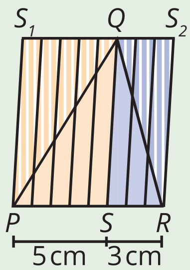


Likewise we can duplicate $\triangle SQR$ and rotate it to form parallelogram SQS_2R.

The resulting parallelogram S_1S_2RP has double the area of $\triangle PQR$, **2** × **36** cm^2 = **72** cm^2.



S_1S_2RP can be split into eight identical parallelograms, each with base length **1** cm.

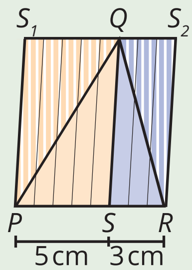


We can see that SQS_2R is three-eighths of the area of S_1S_2RP.

$\frac{3}{8}$ × **72** cm^2 = **27** cm^2.

$\triangle SQR$ is half of the area of SQS_2R.

The area of $\triangle SQR$ is **27** cm^2 ÷ **2** = **13.5** cm^2.



FOLLOW-UP: *Point T lies on the line that passes through points P and R . If the area of △PTQ is 45 cm², find all possible values for the length RT.* [2 cm and 18 cm]

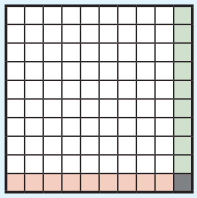
4C. The question is: Express, as a decimal, the difference between $\sqrt{0.01}$ and $\frac{0.013}{0.01}$.

METHOD 1: *Find equivalent fractions.*

We begin by considering the value of **0.01**.

Tens	Ones	Tenths	Hundredths
	0	0	1

0.01 is equivalent to one hundredth, or $\frac{1}{100}$.



For $\sqrt{0.01}$, we need to find a value that, when squared, will equal $\frac{1}{100}$.

$\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$, so $\sqrt{0.01} = \frac{1}{10}$.

(Note that, by convention, $\sqrt{}$ refers to the positive square root.)

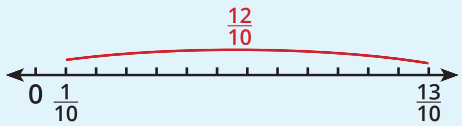
To find the value of $\frac{0.013}{0.01}$, we can consider equivalent fractions.

$$\frac{0.013}{0.01} = \frac{1000}{1000} \times \frac{0.013}{0.01}$$

$$= \frac{13}{10}.$$

The difference between $\frac{1}{10}$ and $\frac{13}{10}$ is $\frac{13}{10} - \frac{1}{10} = \frac{12}{10}$.

As a decimal, the difference between $\sqrt{0.01}$ and $\frac{0.013}{0.01}$ is **1.2**.



METHOD 2: *Reason algebraically.*

Let $\frac{a}{b} = \sqrt{0.01}$, where a and b are both positive.

Then:	$\frac{a}{b} \times \frac{a}{b} = 0.01$
	$\frac{a^2}{b^2} = 0.01$
Multiplying both sides by b^2:	$a^2 = 0.01b^2$
Multiplying both sides by **100**:	$100a^2 = b^2$
Square root both sides:	$10a = b$
Dividing both sides by b:	$10 \times \frac{a}{b} = 1$
Dividing both sides by **10**:	$\frac{a}{b} = \frac{1}{10}$.

Therefore, $\sqrt{0.01} = \frac{1}{10} = 0.1$.

Let $\frac{c}{d} = \frac{0.013}{0.01}$.

Multiplying both sides by **0.01**:	$0.01\frac{c}{d} = 0.013$
Multiplying both sides by d:	$0.01c = 0.013d$
Multiplying both sides by **1000**:	$10c = 13d$
Dividing both sides by **10**:	$c = 1.3d$
Dividing both sides by d:	$\frac{c}{d} = 1.3$

Therefore, $\frac{0.013}{0.01} = 1.3$.

As a decimal, the difference between $\sqrt{0.01}$ and $\frac{0.013}{0.01}$ is **1.3 – 0.1 = 1.2**.

FOLLOW-UP: *Express, as a decimal, the difference between* $\sqrt{0.0001}$ *and* $\frac{0.013}{0.0001}$. [129.99]

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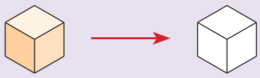
4D. The question is: How many of the unit cubes have exactly one face painted white?

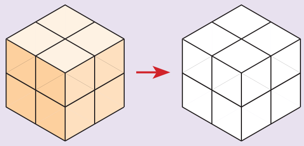
METHOD: *Divide a complex shape, and find a pattern.*

A single **1** cm × **1** cm × **1** cm cube is a unit cube.

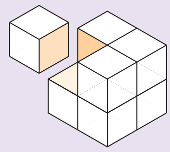


A single unit cube will have all **6** faces painted white.

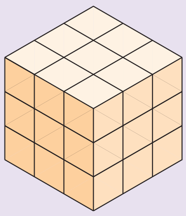
A **2** cm × **2** cm × **2** cm cube can be cut into eight unit cubes.



Each unit cube will have **3** painted faces.

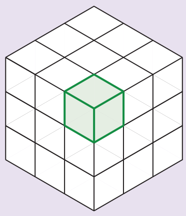


A **3** cm × **3** cm × **3** cm cube can be cut into **27** unit cubes.



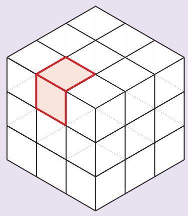
A unit cube at a vertex will have **3** painted faces.

A cube has **8** vertices.



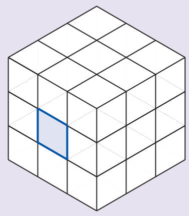
A unit cube on an edge, but not on a vertex, has **2** painted faces.

A cube has **12** edges.



A unit cube on a face, but not on an edge, will have **1** painted face.

A cube has **6** faces.

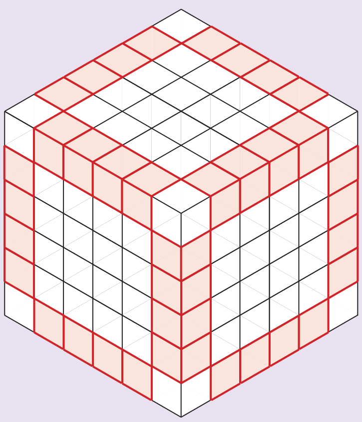


After being cut, the wooden cube yields 48 unit cubes that have exactly 2 painted faces.

These unit cubes must be located on an edge, but not on a vertex.

Since a cube has 12 edges, there would be 48 ÷ 12 = 4 such unit cubes per edge.

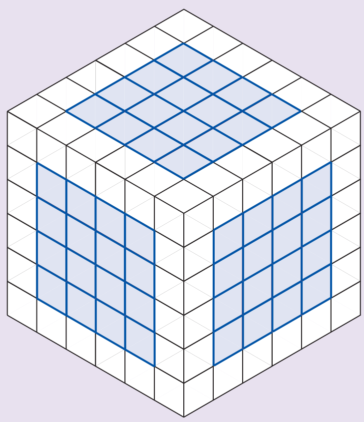
This would occur if the original wooden cube had a side length of **6** cm.



On a cube with side length **6** cm, each face will yield **4 × 4 = 16** unit cubes that each have exactly **1** painted face.

There are 16 of these unit cubes on each face.

With **6** faces on a cube, there are **6 × 16 = 96** unit cubes that each have exactly one painted face.



Follow-Up: *If there had been seventy-two unit cubes with exactly two faces painted white, how many unit cubes would have exactly one face painted white?* [216]

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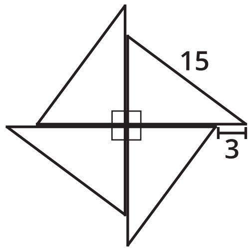
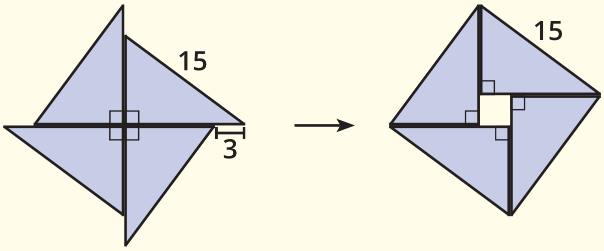
4E. The question is: Find the total area of the star pattern, in square cm.



METHOD 1: *Convert to a more convenient form.*

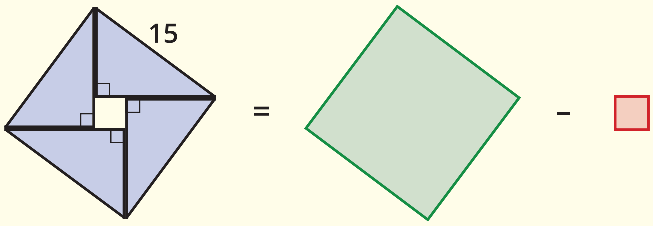
We can begin by re-aligning the triangles, as shown.



The resulting square has a side length of 15 cm, and an area of 15 cm × 15 cm = 225 cm^2.

The hole in the middle has a side length of 3 cm, and an area of 3 cm × 3 cm = 9 cm^2.

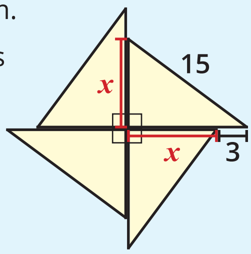
The area of the four triangles is therefore 225 cm^2 – 9 cm^2 = **216** cm^2.



METHOD 2: *Use Pythagoras' Theorem, and reason algebraically.*

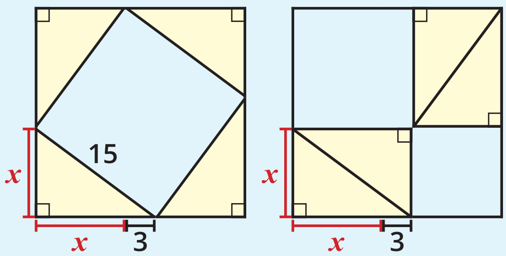
Let the short side of the triangle have length x cm.

The side lengths for one triangle are x cm, $(x+3)$ cm, and 15 cm.



Pythagoras' Theorem states that the square of the hypotenuse is equal to the sum of the squares of the other two sides.

The diagrams demonstrate one of the proofs of this result.



Using Pythagoras' Theorem, we have: $x^2 + (x+3)^2 = 15^2 = 225$

This can be solved by factorising:

$2x^2 + 6x - 216 = 0$

$2(x+12)(x-9) = 0$

$x = 9$

Note that, since x represents a length, $x \neq -12$.

It can also be solved by Guess, Check and Refine:

x	7	8	9
x^2	49	64	81
$(x+3)^2$	100	121	144
$x^2 + (x+3)^2$	149	185	225

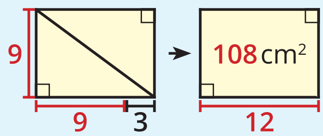
The conditions are met when $x = 9$.

Another way is to check if the triangle sides are in the Pythagorean triad ratio 3:4:5.

3:4:5 = 9:12:15.

Since $9^2 + 12^2 = 15^2$, the conditions are met when $x = 9$.

When $x = 9$, the area of 2 triangles is 9 cm × 12 cm = 108 cm^2.



The area of 4 triangles is therefore 2 × 108 cm^2 = **216** cm^2.

FOLLOW-UP: *Find replacement numbers for the 15 and 3 in the original problem so that the area of the star pattern would be 375.* [One possible solution: Replace 15 with 20, and replace 3 with 5]