2024 Maths Olympiads Division S Preparation Kit

Preparing for the APSMO Maths Olympiads

The purpose of this Preparation Kit is to provide students with an opportunity to familiarise themselves with the concepts, and terminology, that will subsequently be used in the four competition papers for 2024.

For each of the problems in this kit, a number of different solution methods are suggested, so that students can be exposed to multiple ways of approaching mathematical problems.

The kit additionally includes an updated reference sheet for relevant skills and terminology. This reference sheet can also be found in the Resources section of your Members Portal.

Examples of how this kit may be used include:

- Reinforcing previously learned concepts and terminology
- Introducing new or different solution methods
- Providing diagrams that support a teacher's or student's explanations
- Offering problem-solving homework
- Supporting students' own study as a standalone resource

Further questions and solution methods can also be found in the APSMO resource books, available from www.apsmo.edu.au.

A. In the given cryptarithm, different letters represent different digits and no leading digit equals zero.

Find the greatest possible whole number value for $YARD$.

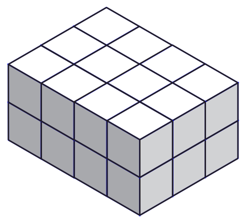
$$\begin{array}{r} F\ O\ O\ T \\ F\ O\ O\ T \\ +\ F\ O\ O\ T \\ \hline Y\ A\ R\ D \end{array}$$

Write your answers in the boxes on the back.

← *Keep your answers hidden by folding backwards on this line.*

B. How many different rectangular prisms can be constructed using exactly 24 identical unit cubes?

[One example is shown with dimensions 2 × 3 × 4.]



C. Students in the school band each play one instrument: woodwind, brass, or percussion.

20 students play woodwind instruments.

28 students play brass instruments.

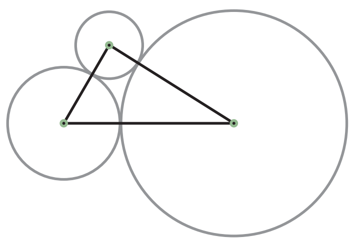
One fifth of the students play percussion instruments.

How many students are there in the school band?

D. Three circles, with areas $9\pi\,\text{cm}^2$, $25\pi\,\text{cm}^2$ and $100\pi\,\text{cm}^2$, are externally tangent as shown.

A triangle is formed by connecting the three centres of the circles.

Find the perimeter of the triangle, in centimetres.



The area of a circle is equal to πr^2, where r is the radius of the circle. π is approximately equal to 3.14 or $\frac{22}{7}$.

E. $\sqrt[2]{16} = 4$, because $4^2 = 4 \times 4 = 16$.

By convention, $\sqrt[2]{\ }$ refers to the positive square root.

$\sqrt[3]{27} = 3$, because $3^3 = 3 \times 3 \times 3 = 27$.

$\sqrt[4]{16} = 2$, because $2^4 = 2 \times 2 \times 2 \times 2 = 16$.

Express $\sqrt[4]{\sqrt[3]{\sqrt[2]{2^{24}}}}$ as a whole number.

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A.	*Fold here. Keep your answers hidden.*
B.	
C.	
D.	
E.	

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F. Calculate: $\dfrac{13^2 - 12^2}{13 + 12}$

G. A prime number is a counting number with exactly two factors, 1 and itself.

The length and width of a rectangle are both prime numbers of centimetres.

The rectangle has a perimeter of 30 centimetres.

What is the area of the rectangle, in square centimetres?

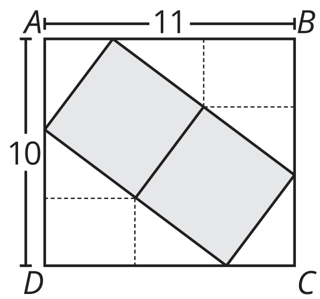
H. 2^n means that 2 is multiplied by itself n times.
For example, 2^2 means $2 \times 2 = 4$.
Since the ones digit of 4 is 4, we say that 2^2 has 4 in the ones place.
Likewise, 2^4 means $2 \times 2 \times 2 \times 2 = 16$.
The ones digit of 16 is 6, so 2^4 has 6 in the ones place.
2^{20} is 2 multiplied by itself 20 times.
What is the ones digit in the value of 2^{20}?

I. The diagram shows two identical squares (shaded), which share a common side.

The two squares intersect rectangle *ABCD* at four points, as shown.

AB = 11 cm and *AD* = 10 cm.

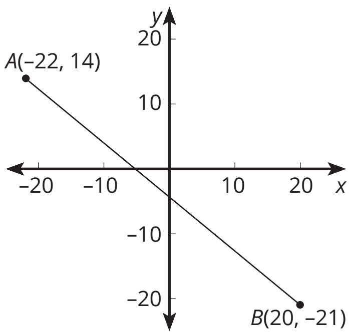
Find the number of square centimetres in the combined area of the two squares.



J. A lattice point is a point on a Cartesian plane where both the x and y co-ordinates are integers.

The interval $\overline{AB}$ joins the points $A(-22, 14)$ and $B(20, -21)$.

Including endpoints A and B, how many lattice points lie on $\overline{AB}$?



Write your answers in the boxes on the back.

Keep your answers hidden by folding backwards on this line.

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F.	*Fold here. Keep your answers hidden.*
G.	
H.	
I.	
J.	

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Example Solution A

In the given cryptarithm, different letters represent different digits and no leading digit equals zero.

Find the greatest possible whole number value for *Y A R D.*

	F	*O*	*O*	*T*
	F	*O*	*O*	*T*
+	*F*	*O*	*O*	*T*
	Y	*A*	*R*	*D*

Strategy 1: Use Reasoning to Determine Each Digit

The sum of the three 4-digit numbers is a 4-digit number.

The greatest value for ***F*** is **3** and ***Y*** = **9**.

	3	*O*	*O*	*T*
	3	*O*	*O*	*T*
+	**3**	*O*	*O*	*T*
	9	*A*	*R*	*D*

Since the sum ***O*** + ***O*** + ***O*** is a 1-digit sum, and since **3** is already assigned to ***F***, the greatest value for ***O*** is **2**.

	3	**2**	**2**	*T*
	3	**2**	**2**	*T*
+	3	**2**	**2**	*T*
	9	*A*	*R*	*D*

In order to maximise ***YARD***, ***T*** needs to be as great as possible.

Since ***Y*** = **9**, ***T*** ≠ **9** so try ***T*** = **8**.

Then ***D*** = **4** and ***R*** = **8** which is not possible since it must be different from ***T***.

	3	2	2	**8**
	3	2	2	**8**
+	3	2	2	**8**
	9	*A*	*R*	*D*

Try ***T*** = **7**. Then ***D*** = **1**, ***R*** = **8**, and ***A*** = **6**.

This results in the maximum value for ***YARD*** which is **9681**.

	3	2	2	**7**
	3	2	2	**7**
+	3	2	2	**7**
	9	6	8	1

Strategy 2: Try Making *YARD* as Great as Possible

The maximum value for the 4-digit number ***YARD*** is **9876** .

Unfortunately, this cannot occur since if ***Y*** = **9**, ***F*** = **3**, and ***O*** must be less than **3** in order to not require a regrouping in the hundreds position.

	3	*O*	*O*	*T*
	3	*O*	*O*	*T*
+	**3**	*O*	*O*	*T*
	9	**8**	**7**	**6**

Therefore, the greatest value for ***O*** is **2**.

This leads to ***A*** = **6**.

When ***O*** = **2**, ***R*** equals **6**, **7**, or **8** depending on the regrouping in the ones column.

	3	**2**	**2**	*T*
	3	**2**	**2**	*T*
+	3	**2**	**2**	*T*
	9	*A*	*R*	*D*

Since we want the greatest value for ***YARD***, we want ***R*** = **8** which can occur when ***T*** is **7**, **8**, or **9**.

8 and **9** are already assigned to letters, so let ***T*** = **7** and ***D*** = **1**.

Therefore, the greatest value for *YARD* is **9681**.

	3	2	2	**7**
	3	2	2	**7**
+	3	2	2	**7**
	9	6	8	1

Answer: 9681

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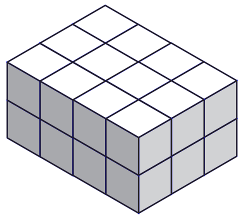
Example Solution B



How many different rectangular prisms can be constructed using exactly 24 identical unit cubes?

[One example is shown with dimensions 2 × 3 × 4.]

Strategy: List All Possible Cases

The three dimensions of the rectangular prism must be positive integers that multiply to **24**.

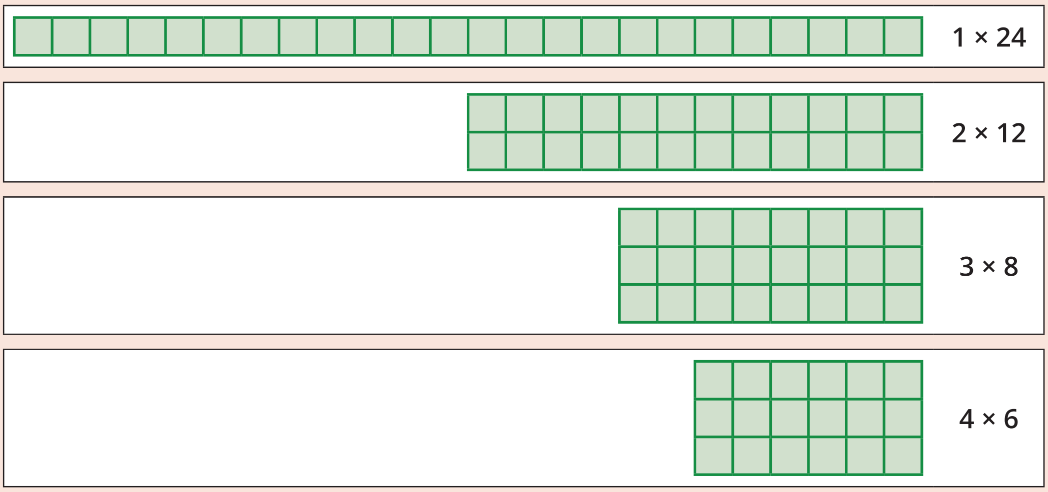
The factors of **24** are **1**, **2**, **3**, **4**, **6**, **8**, **12**, and **24**.

Case 1: Shortest dimension is 1 unit long.

The product of the other two dimensions must be **24 ÷ 1 = 24**.

With a shortest dimension of **1** unit, the possible dimensions for the rectangular prism are:

- 1 × 1 × 24,
- 1 × 2 × 12,
- 1 × 3 × 8, and
- 1 × 4 × 6.

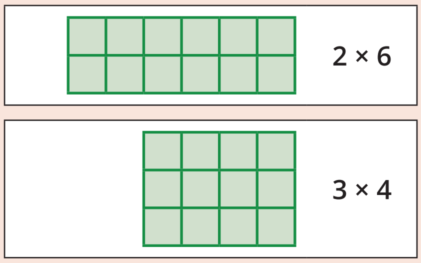


Case 2: Shortest dimension is 2 units long.

The product of the other two dimensions must be **24 ÷ 2 = 12**.

With a shortest dimension of **2** units, the possible dimensions for the rectangular prism are:

- 2 × 2 × 6, and
- 2 × 3 × 4.



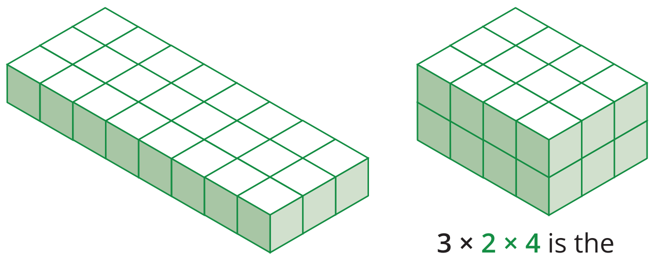
Case 3: Shortest dimension is 3 units long.

The product of the other two dimensions must be **24 ÷ 3 = 8**.

If one dimension is **3** units long, the possible dimensions for the rectangular prism are **3** × 1 × 8, and **3** × 2 × 4.

In both cases, one of the other dimensions is shorter than **3** units.

We counted those prisms already, when we listed prisms with a shortest dimension of **1** or **2** units.



3 × 1 × 8 is the same prism as **1** × 3 × 8.

3 × 2 × 4 is the same prism as **2** × 3 × 4.

We cannot construct a rectangular prism using **24** unit cubes, where the shortest dimension is **3** units long.

All together, there are **4 + 2 = 6** possible rectangular prisms constructible with **24** unit cubes.

Answer: 6

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Example Solution C

Students in the school band each play one instrument: woodwind, brass, or percussion.

20 students play woodwind instruments.

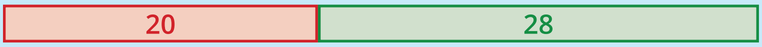
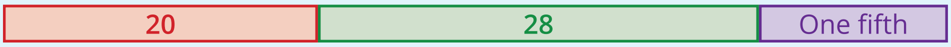
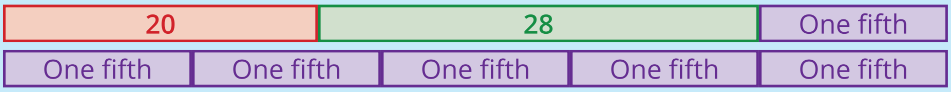
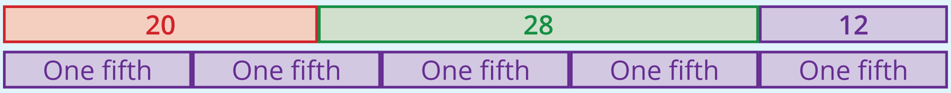
28 students play brass instruments.

One fifth of the students play percussion instruments.

How many students are there in the school band?

Strategy: Work Backwards

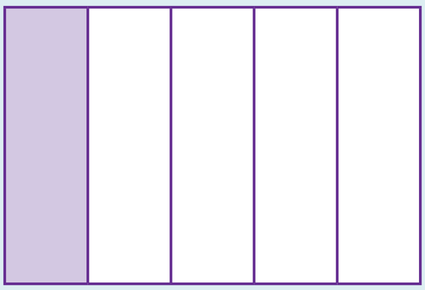
Method 1: Draw a Bar Diagram

In the school band, 20 students play woodwind instruments.	
28 students play brass instruments.	
One-fifth of the students play percussion instruments.	
If one-fifth of the students play percussion instruments, then four-fifths of the students must play woodwind or brass.	
Four-fifths of the students is equivalent to 20 + 28 = 48 students, so one-fifth must be 48 ÷ 4 = 12 students.	

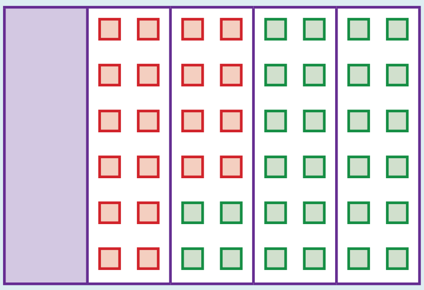
There are 20 + 28 + 12 = **60** students in the band.

Method 2: Consider the fraction of students who play woodwind and brass.

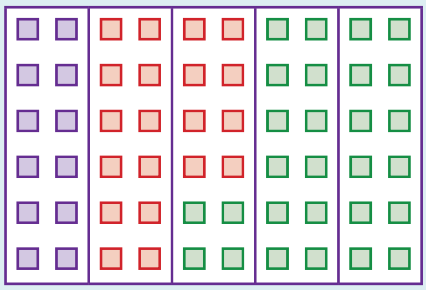
One-fifth of the students play percussion instruments.



With 20 woodwind and 28 brass, there are 20 + 28 = 48 other students in the other four-fifths of the band.



With **48** students in four-fifths of the band, one-fifth of the band comprises **48** ÷ 4 = 12 students.



There are 5 × 12 = **60** students in the band.

Answer: 60

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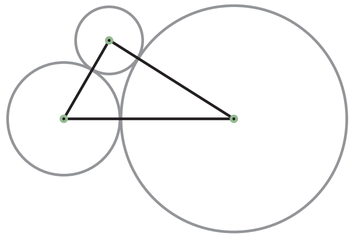
Example Solution D

Three circles, with areas $9\pi\,cm^2$, $25\pi\,cm^2$ and $100\pi\,cm^2$, are externally tangent as shown.

A triangle is formed by connecting the three centres of the circles.

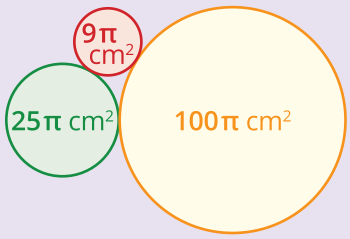
Find the perimeter of the triangle, in centimetres.

The area of a circle is equal to πr^2, where r is the radius of the circle. π is approximately equal to 3.14 or $\frac{22}{7}$.



Strategy: Draw a Diagram, and Reason Algebraically

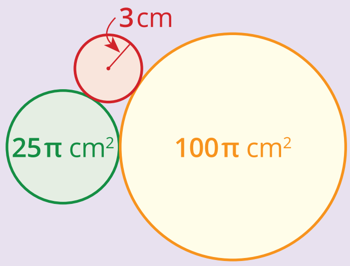
The three circles have areas $9\pi\,cm^2$, $25\pi\,cm^2$ and $100\pi\,cm^2$.



The area of a circle is equal to πr^2, where r is the radius of the circle.

For a circle with area $9\pi\,cm^2$:

	$\pi r^2 = 9\pi\,cm^2$
Divide both sides by π:	$r^2 = 9\,cm^2$
Square root both sides:	$r = 3\,cm$

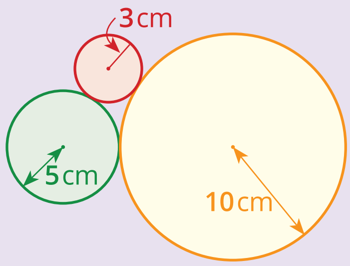


The radius of that circle would be 3 cm.

Similarly, for circles with areas $25\pi\,cm^2$ and $100\pi\,cm^2$:

	$\pi r^2 = 25\pi\,cm^2$
Divide both sides by π:	$r^2 = 25\,cm^2$
Square root both sides:	$r = 5\,cm$

	$\pi r^2 = 100\pi\,cm^2$
Divide both sides by π:	$r^2 = 100\,cm^2$
Square root both sides:	$r = 10\,cm$



The radius of the $25\pi\,cm^2$ circle would be 5 cm.

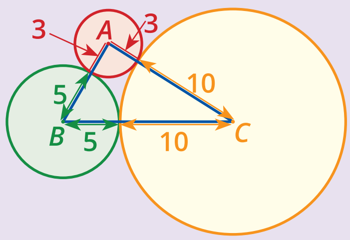
The radius of the $100\pi\,cm^2$ circle would be 10 cm.

Let's label the centres of the circles *A*, *B* and *C*.

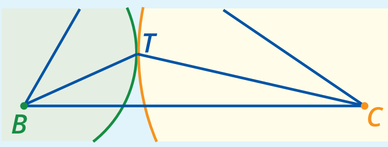
We want to find the perimeter of triangle *ABC*.

We can see that each side of triangle *ABC* passes through the point where two circles touch.* *(see below for explanation)*

Therefore, the perimeter of triangle *ABC* is (3 + 5) + (5 + 10) + (10 + 3) = 36 cm.

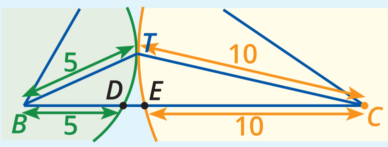


* The diagrams at the right illustrate what would happen if (say) the straight line *BC* did ***not*** pass through the tangent point, here labelled *T*.



We can find a different point ***D***, on the circumference of the circle with centre *B*, which lies on the line *BC*. Since *B* is the centre, *BD* would be a radius with length 5 cm.

Similarly, we can find a different point ***E***, on the circumference of the circle with centre *C*, which lies on the line *BC*. Since *C* is the centre, *CE* would be a radius with length 10 cm.



Since there is also an interval *DE* with a non-zero length, *BC* will have length 5 cm + 10 cm + [length of *DE*].

This is impossible, since the other sides of triangle *BTC* are 5 cm and 10 cm respectively.

Therefore, *BC* must pass through the tangent point.

Answer: 36 (cm)

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Example Solution E

$\sqrt[2]{16} = 4$, because $4^2 = 4 \times 4 = 16$. By convention, $\sqrt[2]{\ }$ refers to the positive square root.

$\sqrt[3]{27} = 3$, because $3^3 = 3 \times 3 \times 3 = 27$. $\sqrt[4]{16} = 2$, because $2^4 = 2 \times 2 \times 2 \times 2 = 16$.

Express $\sqrt[4]{\sqrt[3]{\sqrt[2]{2^{24}}}}$ as a whole number.

Strategy 1: Work Backwards

Since $2^4 = \underbrace{2 \times 2 \times 2 \times 2}_{\text{4 times}}$,

we know that $2^{24} = \underbrace{2 \times 2}_{\text{24 times}}$.

We begin with the innermost term: the $\sqrt[2]{2^{24}}$ part from $\sqrt[4]{\sqrt[3]{\sqrt[2]{2^{24}}}}$.

We are given that $\sqrt[2]{16} = 4$, because $4^2 = 4 \times 4 = 16$.

Since $2^{24} = \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}_{\text{12 times}} \times \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}_{\text{12 times}}$,

$= 2^{12} \times 2^{12}$,

$\sqrt[2]{2^{24}} = 2^{12}$.

Substituting 2^{12} for $\sqrt[2]{2^{24}}$ in the expression, we have: $\sqrt[4]{\sqrt[3]{2^{12}}}$.

We are given that $\sqrt[3]{27} = 3$, because $3^3 = 3 \times 3 \times 3 = 27$.

Since $2^{12} = \underbrace{2 \times 2 \times 2 \times 2}_{\text{4 times}} \times \underbrace{2 \times 2 \times 2 \times 2}_{\text{4 times}} \times \underbrace{2 \times 2 \times 2 \times 2}_{\text{4 times}}$,

$= 2^4 \times 2^4 \times 2^4$,

$\sqrt[3]{2^{12}} = 2^4$.

We now have $\sqrt[4]{2^4}$.

$\sqrt[4]{16} = 2$, because $2^4 = 2 \times 2 \times 2 \times 2 = 16$.

Since $2^4 = 2 \times 2 \times 2 \times 2$,

$\sqrt[4]{2^4} = 2$.

Therefore $\sqrt[4]{\sqrt[3]{\sqrt[2]{2^{24}}}} = 2$.

Strategy 2: Reason Algebraically

Let $\sqrt[4]{\sqrt[3]{\sqrt[2]{2^{24}}}} = x$.

Raising both sides to the power of 4: $\sqrt[3]{\sqrt[2]{2^{24}}} = x^4$.

Raising both sides to the power of 3: $\sqrt[2]{2^{24}} = x^4 \times x^4 \times x^4$

$= (x \times x \times x \times x) \times (x \times x \times x \times x) \times (x \times x \times x \times x)$

$= x^{12}$.

Raising both sides to the power of 2: $2^{24} = x^{12} \times x^{12}$

$= \underbrace{(x \times x \times x \times \ldots \times x)}_{\text{12 times}} \times \underbrace{(x \times x \times x \times \ldots \times x)}_{\text{12 times}}$

$= x^{24}$.

Since $x^{24} = 2^{24}$, we can see that $x = 2$.

Answer: 2

Example Solution F

Calculate: $\dfrac{13^2 - 12^2}{13 + 12}$

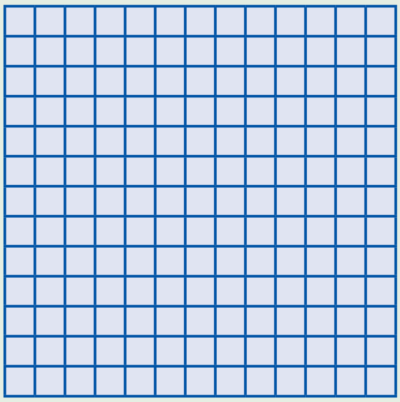
Strategy 1: Perform the Calculation

Since **13 × 13 = 169**, and **12 × 12 = 144**, we have:

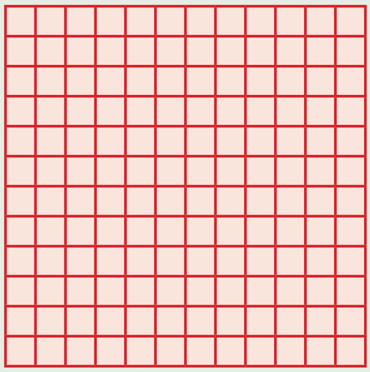
$$\frac{13^2 - 12^2}{13 + 12} = \frac{169 - 144}{13 + 12} = \frac{25}{25} = \mathbf{1}.$$

Strategy 2: Draw a Diagram

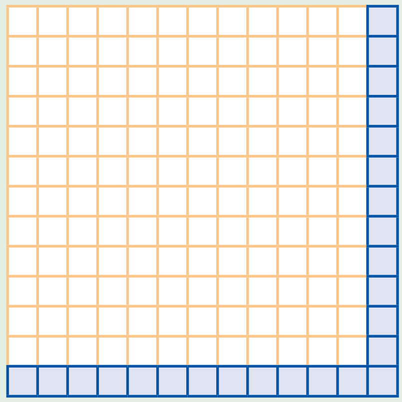
We can represent 13^2 as an array with **13** columns and **13** rows.



Likewise, 12^2 can be represented as an array with **12** columns and **12** rows.



So $13^2 - 12^2$ is equivalent to removing the **12 × 12** array from the **13 × 13** array.



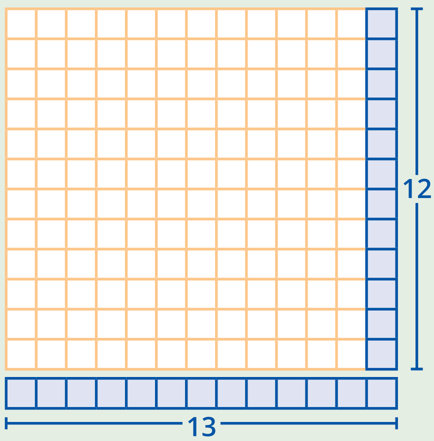
After removing the **12 × 12** array, what remains can be thought of as

- a column of **12**

and

- a row of **13**,

for a total of **12 + 13 = 25**.



Therefore,

$$13^2 - 12^2 = 12 + 13 = 25$$

and so

$$\frac{13^2 - 12^2}{13 + 12} = \frac{25}{25} = \mathbf{1}.$$

* Note: A more sophisticated version of this solution would see the students recognising that **13^2 – 12^2** is the difference of two squares, and so can be factorised.

$$13^2 - 12^2 = (13 + 12)(13 - 12)$$

$$\frac{13^2 - 12^2}{13 + 12} = \frac{(13 + 12)(13 - 12)}{13 + 12}$$

$$= 13 - 12.$$

Answer: 1

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Example Solution G

A prime number is a counting number with exactly two factors, 1 and itself.

The length and width of a rectangle are both prime numbers of centimetres.

The rectangle has a perimeter of 30 centimetres. What is the area of the rectangle, in square centimetres?

Strategy 1: Make an Organised List

Since a prime number is only divisible by **1** and itself, it can only be written as a product of two whole numbers if those whole numbers are **1** and itself.

We can use this idea to list all of the prime numbers.

The first prime number is 2, because its only factors are **1** and **2**. Any further multiples of 2 are not prime.

1	2	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~	11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~	21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~

We can see that the next prime number must be 3. We can now eliminate further multiples of 3.

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~	~~21~~	~~22~~	23	~~24~~	25	~~26~~	~~27~~	~~28~~	29	~~30~~

The next prime number must be 5. We can now eliminate further multiples of 5.

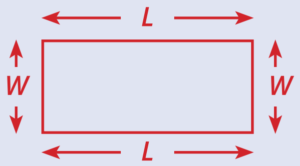
1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~	~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~

Continuing with this method (called Eratosthenes' Sieve), we would find that the prime numbers less than **30** are 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29.

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~	~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~

Let *L* be the length of the rectangle in cm, and *W* be the width of the rectangle in cm.

The perimeter of the rectangle is *L* + *W* + *L* + *W*, as shown.



The problem states that the perimeter of the rectangle is 30 cm.

$L + W + L + W = 30$,

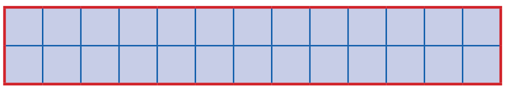
so $L + W = 30 \div 2$

$= 15$.

We can try different prime numbers for the *L* measurement, to see if the *W* is likewise prime.

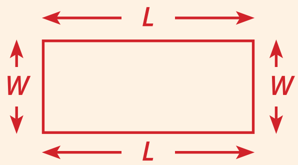
L	*W* = 15 – *L*	Both prime?
2	15 – 2 = 13	Yes
3	15 – 3 = 12	No
5	15 – 5 = 10	No
7	15 – 7 = 8	No
11	15 – 11 = 4	No
13	15 – 13 = 2	Yes

Since the rectangle is either 2 cm long and 13 cm wide, or 13 cm long and 2 cm wide, its area must be 13 cm × 2 cm = **26** cm^2.



Strategy 2: Consider Properties of Numbers

As noted in Strategy 1, the perimeter of a rectangle is equal to *L* + *W* + *L* + *W*.



Therefore,

$L + W + L + W = 30$ cm

and so $L + W = 15$ cm.

Since **15** is odd, *L* and *W* must have different parity (they cannot both be odd, or both even).

2 is the only prime number that is also even, so one of the side lengths must be 2 cm.

The other side length would then be 15 – 2 = 13 cm.

Therefore, the area of the rectangle must be 2 cm × 13 cm = **26** cm^2.

Answer: 26 (cm^2)

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Example Solution H

2^n means that 2 is multiplied by itself n times. For example, 2^2 means 2 × 2 = 4. Since the ones digit of 4 is 4, we can say that 2^2 has 4 in the ones place.

Likewise, 2^4 means 2 × 2 × 2 × 2 = 16. The ones digit of 16 is 6, so 2^4 has 6 in the ones place.

2^{20} is 2 multiplied by itself 20 times. What is the ones digit in the value of 2^{20}?

Multiplying out 2^{20} looks very long and complicated. However, we only need to find the ones digit.

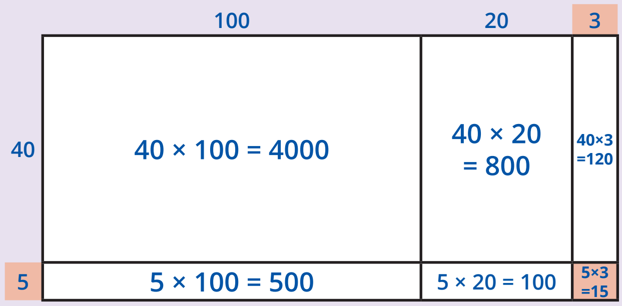
Can we solve this problem without actually finding the value of 2^{20}?

Let's consider what happens when we multiply large numbers - say, 123 × 45.

Whether we use a written algorithm, or the area model, we can see that:

- the only part of 123 that affects the ones digit of the result is the 3, and
- the only part of 45 that affects the ones digit of the result is the 5.

```
    1 2 3
  ×   4 5
  -------
    6 1 5
  4 9 2 0
  -------
  5 5 3 5
```



The other place values represent multiples of 10, and so they will have no effect on the ones digit of the result.

Strategy 1: Multiply out 2^{20}

Since 2^4 = 2 × 2 × 2 × 2 = 16,
then 2^5 = 2 × 2 × 2 × 2 × 2
= 16 × 2
= 32.

So the ones digit of 2^5 is 2.

Note that, instead of
16 × 2 = 32,
we could have worked out
6 × 2 = 12.

The ones digit in the result will be the same.

2^{10} = 2×2×2×2×2 × 2×2×2×2×2 (10 times)
= 2×2×2×2×2 (5 times) × 2×2×2×2×2 (5 times)
= 2^5 × 2^5
= 32 × 32
= 1024.

So 2^{10} has 4 in the ones place.

Again, because 2^5 has 2 in the ones place, we could just have worked out 2 × 2 = 4.

2^{20} = 2 × 2 × ... × ... × 2 × 2 (20 times)
= 2 × 2 × ... × 2 (10 times) × 2 × ... × 2 (10 times)
= 2^{10} × 2^{10}
= 1024 × 1024.

To find the ones digit of the product, we don't need to multiply it all out.

Since 2^{20} = 2^{10} × 2^{10}, the ones digit for 2^{20} is the same as the ones digit for 4 × 4 = 16.

Therefore the ones digit of 2^{20} is 6.

Strategy 2: Find a Pattern

When we list the values for 2^1, 2^2, 2^3, 2^4 and so on, the ones digits appear to occur in the order 2, 4, 8, 6, repeat.

With a four-digit pattern, every fourth position will have the same digit.

Therefore the ones digit for 2^4, 2^8, 2^{12}, and every 2^n where n is a multiple of 4, would have the same digit.

Since 20 is a multiple of 4, from the pattern we can reason that the ones digit of 2^{20} will also be 6.

	Value	Ones digit
2^1 = 2	2	2
2^2 = 2 × 2	4	4
2^3 = 2 × 2 × 2	8	8
2^4 = 2 × 2 × 2 × 2	16	6
2^5 = 2 × 2 × 2 × 2 × 2	32	2
2^6 = 2 × 2 × 2 × 2 × 2 × 2	64	4
2^7 = 2 × 2 × 2 × 2 × 2 × 2 × 2	128	8
2^8 = 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2	256	6
2^9 = 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2	512	2
2^{10} = 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2	1024	4
2^{11} = 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2	2048	8
2^{12} = 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2	4096	6

Answer: 6

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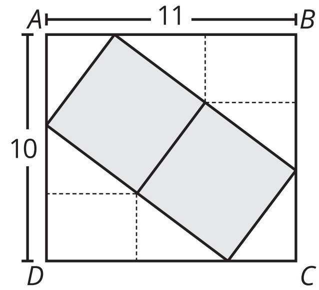
Example Solution I

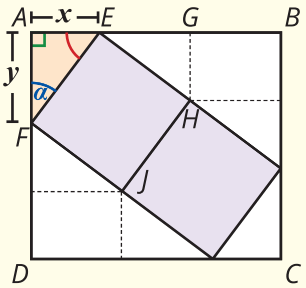
The diagram shows two identical squares (shaded), which share a common side. The two squares intersect rectangle *ABCD* at four points, as shown. *AB* = 11 cm and *AD* = 10 cm.

Find the number of square centimetres in the combined areas of the two squares.



Strategy: Identify Congruent Triangles

We begin by considering the marked triangle, △*AEF*.

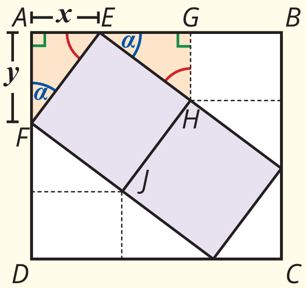


Let ∠*AFE* = α°.

The angle sum of a triangle is 180°, and ∠*EAF* = 90°.

Therefore, ∠*AEF* = (180 – α)°.

We know that ∠*AEG* = 180°, since it is a straight angle.



EFJH is a square (given).

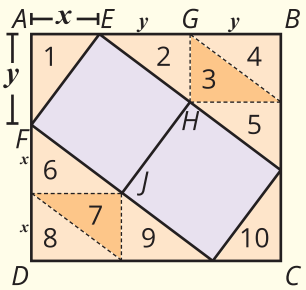
Therefore, ∠*FEH* = 90°,
∠*GEH* = α°, and
∠*GHE* = (180 – α)°.

Since *EFJH* is a square, we also know that *EF* and *EH* are both the same length.

We can see (and use the Angle-Angle-Side congruence test to prove) that △*AEF* and △*GHE* are identical.

This means that both these triangles must have the same side lengths.

There are, in total, **10** congruent triangles surrounding the two squares.



Since the triangles are congruent,

$x + 2y = 11.$ *(Eq. 1, representing the length of AB)*

$2x + y = 10.$ *(Eq. 2, representing the length of AD)*

Method 1: *(Eq. 1)* + *(Eq. 2)*

$x + 2y + 2x + y = 11 + 10$
$3x + 3y = 21$
$x + y = 7.$ *(Eq. 3)*
$2x + y - (x + y) = 10 - 7.$
$x = 3$, and $y = 4$.

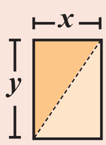
Method 2: 2 × *(Eq. 1)* – *(Eq. 2)*

$2x + 4y = 22$
$2x + y = 10$
$2x + 4y - (2x + y) = 22 - 10$
$3y = 12$
$y = 4$, and $x = 3$.

Method 1: Subtract the Triangles from the Total Area

The area of *ABCD* is 11 cm × 10 cm = 110 cm².

Combining two of the triangles results in a rectangle with area 3 cm × 4 cm = 12 cm².

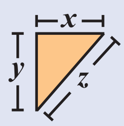


There are 5 pairs of triangles, with a total area of 5 × 12 cm² = 60 cm².

The squares have an area of 110 cm² – 60 cm² = **50** cm².

Method 2: Use Pythagoras' Theorem

By Pythagoras' Theorem, we have $z^2 = x^2 + y^2$, where z represents the side length of one of the squares.



Since $z^2 = 9 + 16 = 25$, we know that the area of one square, $z^2 = 25$ cm².

Therefore the combined areas of both squares is 2 × 25 cm² = **50** cm².

Answer: 50 (cm²)

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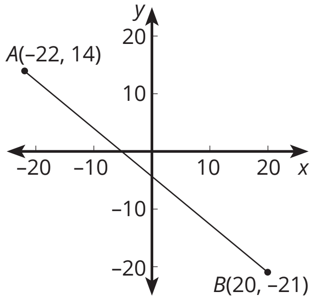
Example Solution J

A lattice point is a point on a Cartesian plane where both the *x* and *y* co-ordinates are integers.

The interval $\overline{AB}$ joins the points *A*(–22, 14) and *B*(20, –21).

Including endpoints *A* and *B*, how many lattice points lie on $\overline{AB}$?

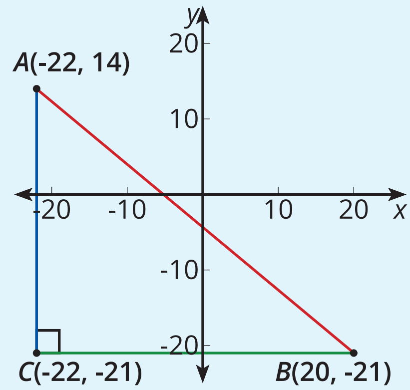


Strategy 1: Consider the Gradient of AB

We begin by constructing the point ***C*** that has the same *x*-value as ***A***, and the same *y*-value as ***B***.

This results in a right-angled triangle ***ABC***.

We can use this triangle to determine the gradient of ***AB***.

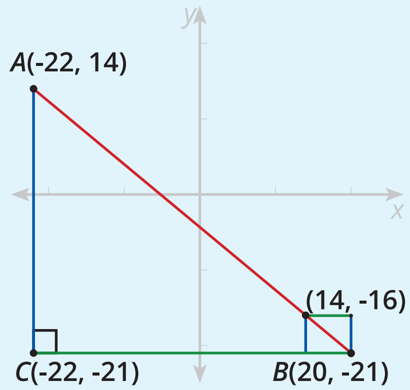


The rise *CA* is 14 – (–21) = 35.

The run *BC* is –22 – 20 = –42.

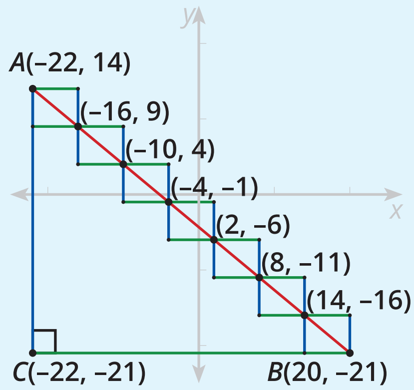
Therefore the gradient of ***AB*** is

$\frac{CA}{BC} = \frac{35}{-42} = -\frac{5}{6}$.



Since the gradient of ***AB*** is $-\frac{5}{6}$, then there will be a lattice point 6 units to the left and 5 units up from ***B***(20, –21).

This means that there is a lattice point at (14, –16).



We can now continue to locate lattice points that are 6 units to the left and 5 units up from the previous lattice point.

We can see that, including *A* and *B*, there are **8** lattice points on $\overline{AB}$.

Strategy 2: Build a Table, and Find a Pattern

The gradient of the line through *A*(-22, 14) and *B*(20, -21) is

$$\frac{(-21) - 14}{20 - (-22)} = -\frac{5}{6}.$$

Since this line passes through *A*(-22, 14), the equation of the line would be:

$$y - 14 = -\frac{5}{6}(x - (-22))$$

$$y = -\frac{5}{6}x - \frac{13}{3}$$

To find lattice points for $-22 \le x \le 20$ on the line $y = -\frac{5}{6}x - \frac{13}{3}$, consider the ***y***-value for every integer value of ***x***.

x	-22	-21	-20	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9
y	14	$13\frac{1}{6}$	$12\frac{1}{3}$	$11\frac{1}{2}$	$10\frac{2}{3}$	$9\frac{5}{6}$	9	$8\frac{1}{6}$	$7\frac{1}{3}$	$6\frac{1}{2}$	$5\frac{2}{3}$	$4\frac{5}{6}$	4	$3\frac{1}{6}$

We can see that there is a pattern. A lattice point is occurring for every 6th integral ***x***-value.

x	-22	...	-16	...	-10	...	-4	...	2	...	8	...	14	...	20
y	14	...	9	...	4	...	-1	...	-6	...	-11	...	-16	...	-21

Therefore there are **8** lattice points on $\overline{AB}$.

Answer: 8

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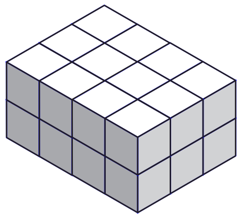
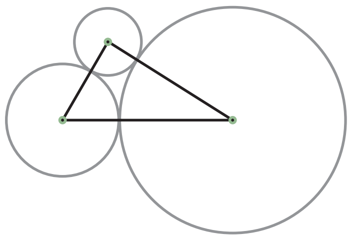
Answers

A. **Answer: 9681**	In the given cryptarithm, different letters represent different digits and no leading digit equals zero. Find the greatest possible whole number value for *Y A R D*. $\begin{array}{cccc} & F & O & O & T \\ & F & O & O & T \\ + & F & O & O & T \\ \hline & Y & A & R & D \end{array}$
B. **Answer: 6**	How many different rectangular prisms can be constructed using exactly 24 identical unit cubes? [One example is shown with dimensions $2 \times 3 \times 4$.] 
C. **Answer: 60**	Students in the school band each play one instrument: woodwind, brass, or percussion. 20 students play woodwind instruments. 28 students play brass instruments. One fifth of the students play percussion instruments. How many students are there in the school band?
D. **Answer: 36** (cm)	Three circles, with areas $9\pi\,cm^2$, $25\pi\,cm^2$ and $100\pi\,cm^2$, are externally tangent as shown. A triangle is formed by connecting the three centres of the circles. Find the perimeter of the triangle, in centimetres.  The area of a circle is equal to πr^2, where r is the radius of the circle. π is approximately equal to 3.14 or $\frac{22}{7}$.
E. **Answer: 2**	$\sqrt[2]{16} = 4$, because $4^2 = 4 \times 4 = 16$. By convention, $\sqrt[2]{}$ refers to the positive square root. $\sqrt[3]{27} = 3$, because $3^3 = 3 \times 3 \times 3 = 27$. $\sqrt[4]{16} = 2$, because $2^4 = 2 \times 2 \times 2 \times 2 = 16$. Express $\sqrt[4]{\sqrt[3]{\sqrt[2]{2^{24}}}}$ as a whole number.

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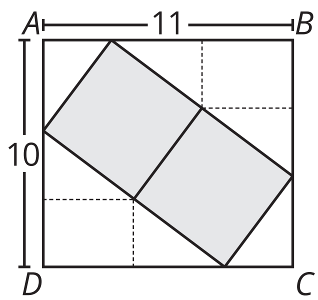
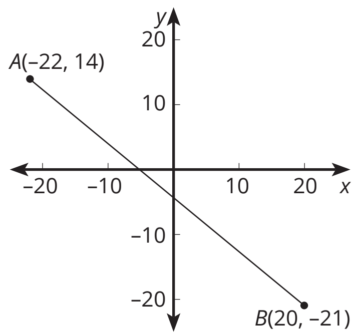
Answers

F. **Answer: 1**	Calculate: $\dfrac{13^2 - 12^2}{13 + 12}$
G. **Answer: 26** (cm^2)	A prime number is a counting number with exactly two factors, 1 and itself. The length and width of a rectangle are both prime numbers of centimetres. The rectangle has a perimeter of 30 centimetres. What is the area of the rectangle, in square centimetres?
H. **Answer: 6**	2^n means that 2 is multiplied by itself n times. For example, 2^2 means $2 \times 2 = 4$. Since the ones digit of 4 is 4, we say that 2^2 has 4 in the ones place. Likewise, 2^4 means $2 \times 2 \times 2 \times 2 = 16$. The ones digit of 16 is 6, so 2^4 has 6 in the ones place. 2^{20} is 2 multiplied by itself 20 times. What is the ones digit in the value of 2^{20}?
I. **Answer: 50** (cm^2)	The diagram shows two identical squares (shaded), which share a common side. The two squares intersect rectangle *ABCD* at four points, as shown. *AB* = 11 cm and *AD* = 10 cm. Find the number of square centimetres in the combined area of the two squares. 
J. **Answer: 8**	A lattice point is a point on a Cartesian plane where both the *x* and *y* co-ordinates are integers. The interval $\overline{AB}$ joins the points *A*(–22, 14) and *B*(20, –21). Including endpoints *A* and *B*, how many lattice points lie on $\overline{AB}$? 

General Knowledge

Basic Terms

• Sum	• Difference	• Product	• Quotient
• Value	• Multiple	• Factor	• Remainder
• Fraction	• Decimal Fraction	• Percentage	• Ratio
• Square / Perfect Square	• Square Root	• Cube / Perfect Cube	• Cube Root

- "Or" is inclusive: "***a*** or ***b***" means "***a*** or ***b*** or both".
- An ellipsis (...) indicates that some information has been omitted intentionally.
  Read "**1 + 2 + 3 + ...**" as "one plus two plus three and so on, without end."
  Read "**1 + 2 + 3 + ... + 10**" as "one plus two plus three and so on up to ten."

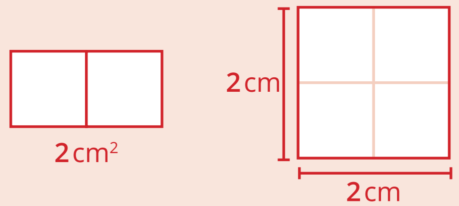
Units of Measurement

Familiarity with units of measurement is assumed, including conversions from one unit to another:

- **Time**: seconds ↔ minutes ↔ hours ↔ days
- **Length**: millimetres ↔ centimetres ↔ metres ↔ kilometres
- **Area**: mm^2 ↔ cm^2 ↔ m^2 ↔ km^2
- **Volume/Capacity**: mm^3 ↔ cm^3 ↔ m^3; millilitres ↔ litres
- **Mass**: grams ↔ kilograms
- **Angles**: degrees (°)

Units of measurement must be correct if given in an answer.

To avoid confusion, read cm^2 as "square centimetres", not "centimetres squared".



2 square centimetres

Possible misinterpretation

Presenting Answers

Unless otherwise specified in a problem, **equivalent numbers or expressions are acceptable**.

- For example, $3\frac{1}{2}$, $\frac{7}{2}$, and 3.5 are equivalent. $3\frac{2}{4}$ and $\frac{70}{20}$ are not in lowest terms and will not be accepted.

After reading a problem, it is useful to indicate the nature of the answer, before commencing the solution strategy. For example:

- "*A* = ____ , *B* = ____ ."
- **"The largest number is ____ ."**
- **"The [sum | difference | product | quotient] is ____ ."**
- **"The probability, as a [fraction | decimal | percentage], is ____ ."**
- **"The perimeter is ____ centimetres."**
- **"The area is ____ square units."**
- **"The average speed is ____ kilometres per hour."**

MATHS OLYMPIAD

Digits and Integers

A **digit** is any one of the ten numerals **0, 1, 2, 3, 4, 5, 6, 7, 8, 9**.

- **358** is a three-digit number.

The **lead digit** (leftmost digit) of a number is not counted as a digit if it is **0**.

- **0358** is a three-digit number.

Terminal zeroes of a number are the zeroes to the right of the last nonzero digit.

- **30 500** has two terminal zeroes.

Whole numbers: { 0, 1, 2, 3, ... }.

Counting numbers, or **Positive Integers**: { 1, 2, 3, ... }.

Integers: { ..., –2, –1, 0, 1, 2, 3, ... }.

- Positive numbers, negative numbers, non-negative numbers, and non-positive numbers are terms that may appear in Division S problems.

Consecutive Numbers are counting numbers that differ by **1**.

- **83, 84, 85, 86, 87.**

Consecutive Even Numbers are multiples of **2** that differ by **2**.

- **36, 38, 40, 42.**

Consecutive Odd Numbers are non-multiples of **2** that differ by **2**.

- **57, 59, 61, 63.**

Factors and Divisibility

Suppose $A = B \times C$, and A, B, and C are all **counting numbers** (1, 2, 3, ...).

- $6 = 2 \times 3$.

Then, **A is divisible by B**, and **A is a multiple of B**.

- **6** is divisible by **2**.
- **6** is a multiple of **2**

Likewise, **A is divisible by C**, and **A is a multiple of C**.

- **6** is divisible by **3**.
- **6** is a multiple of **3**

Both **B and C are factors of A**.

- **2** and **3** are factors of **6**.

A **prime number** is a counting number with exactly two factors, **1** and itself.

- **2, 3, 5, 7, 11, 13, ...**

A **composite number** is a counting number which has at least three different factors.

- **4, 6, 8, 9, 10, 12, ...**

The number **1** is neither prime nor composite since it has exactly one factor.

A number is **factored completely** when it is expressed as a product of only prime numbers.

- $144 = 2 \times 2 \times 2 \times 2 \times 3 \times 3$
  $= 2^4 \times 3^2$.

The **Highest Common Factor (HCF)** of two counting numbers is the largest counting number that divides each of the two numbers, and the remainder is zero.

If the **HCF** of two numbers is **1**, then we say that the numbers are **relatively prime**.

- HCF (12,18) = 6.

The **Lowest Common Multiple (LCM)** of two counting numbers is the smallest number that each of the given numbers divides, and the remainder is zero.

- LCM (12,18) = 36.

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Number

Fractions

For **common** or **simple fractions** $\frac{a}{b}$,

- a (the **numerator**) and b (the **denominator**) are both integers, and
- $b \neq 0$.

In a **unit fraction**, the numerator is 1.	In a **proper fraction**, $a < b$.	In an **improper fraction**, $a \geq b$.
• $\frac{1}{2}$ and $\frac{1}{100}$ are both unit fractions.	• $\frac{1}{2}$ and $\frac{5}{6}$ are both proper fractions.	• $\frac{3}{2}$ and $\frac{11}{8}$ are both improper fractions.

A **complex fraction** is a fraction whose numerator or denominator contains a fraction.

- $\frac{\frac{2}{3}}{5}$, $\frac{2}{\frac{3}{5}}$, $\frac{\frac{2}{3}}{\frac{5}{7}}$, $\frac{2+\frac{3}{5}}{5-\frac{1}{2}}$ are complex fractions.

The fraction $\frac{a}{b}$ is **simplified** (in **lowest terms**) if a and b have no common factor other than **1** - i.e. **HCF(a,b) = 1**.

- $\frac{1}{2}$ and $\frac{3}{2}$ are both expressed in lowest terms. $\frac{2}{4}$ and $\frac{30}{20}$ are not in lowest terms.
- Unless otherwise specified, fraction answers to Olympiad problems must be expressed in lowest terms.

A **decimal** or **decimal fraction** is a fraction whose denominator is a power of ten.

The decimal is written using decimal point notation.

- $0.07 = \frac{7}{100}$, $0.153 = \frac{153}{1000}$, $6.4 = 6\frac{4}{10}$ or $\frac{64}{10}$.

A **recurring decimal**, or **repeating decimal**, is a decimal fraction with a digit, or group of digits, that repeats forever.

- $\frac{1}{3} = 0.333... = 0.\dot{3} = 0.\overline{3}$
- $\frac{1}{6} = 0.1666... = 0.1\dot{6} = 0.1\overline{6}$
- $\frac{1}{7} = 0.142857142857... = 0.\dot{1}4285\dot{7} = 0.\overline{142857}$

A **percentage** is a fraction whose denominator is **100**. The **percent sign** represents the division by **100**.

- $9\% = \frac{9}{100}$, $125\% = \frac{125}{100}$, $0.3\% = \frac{0.3}{100}$ or $\frac{3}{1000}$.

Order of Operations

When an expression has more than one arithmetic symbol, certain operations occur before others.

There are a few ways to remember the order of operations, and mnemonics are often used (e.g. **BIDMAS**; **PEMDAS**).

However, it can also be useful to consider the intent when an arithmetic expression is constructed.

By convention, we observe the following priorities:

1. Perform operations in **parentheses**, **braces**, or **brackets**. The **vinculum** (line in a fraction) is also considered as a grouping symbol, similar to parentheses.
2. Evaluate **exponents** (**indices**).
3. Evaluate **multiplication** and **division**, from left to right.
4. Evaluate **addition** and **subtraction**, from left to right.

Example 1

$30 + 6 \div 2 - 5 \times (9 - 7)$
$= 30 + 6 \div 2 - 5 \times 2$
$= 30 + 3 - 10$
$= 23$

Example 2

$20 - (8 + (1 + 2)^2)$
$= 20 - (8 + 3^2)$
$= 20 - (8 + 9)$
$= 20 - 17$
$= 3$

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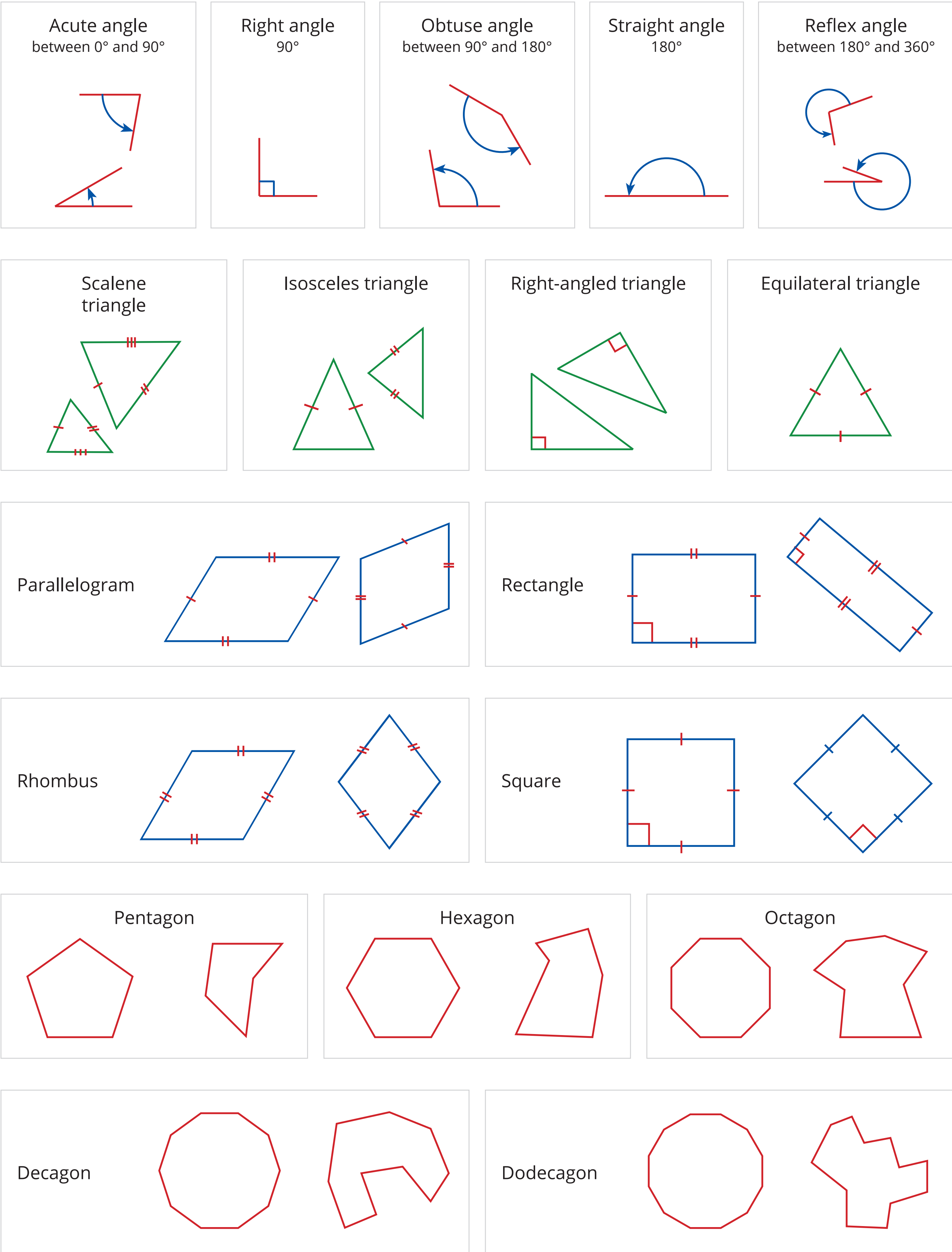
Two-Dimensional Figures



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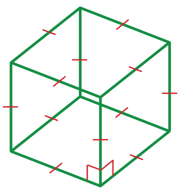
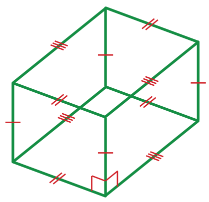
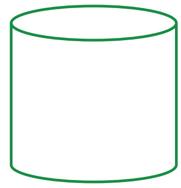
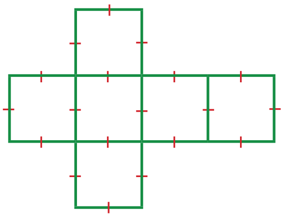
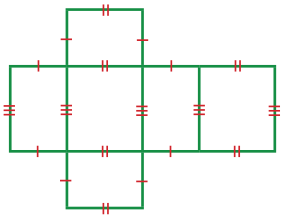
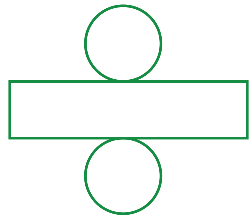
Three-Dimensional Objects

Cube	Rectangular Prism	Right Cylinder
		
One possible net of a cube	One possible net of a rectangular prism	One possible net of a right cylinder
		

Congruence and Similarity

Two geometric figures are **congruent** if they are identical.

- Congruent triangles coincide exactly when one is superimposed upon the other.
- Congruent plane figures have corresponding pairs of sides that are equal, and corresponding pairs of angles that are the same.

Two geometric figures are **similar** if their shape is the same, even though their size may be different.

- All squares are similar, and all circles are similar.

Classification of Geometric Figures

All equilateral triangles are isosceles, but only some isosceles triangles are equilateral.

A square is a rectangle with all sides congruent.

A square is also a rhombus with all angles congruent.

Within the USA/Canada, a trapezium is an irregular quadrilateral.
Outside the USA/Canada, a trapezium is a quadrilateral with at least one pair of parallel sides (known as a “trapezoid” within the USA/Canada).

Calendar Conventions

There was no year **0**. The first century spanned the years **1** to **100** inclusive.

- The **20th century** spanned the years **1901** to **2000** inclusive.
- The **21st century** spans the years **2001** to **2100** inclusive.

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Measures of Centre

The mean, arithmetic mean, or average, of a set of values is

- the sum of the values, divided by
- the number of values.

- For the set **{ 5, 5, 7, 11 }**, the mean is $\frac{5+5+7+11}{4}$ = **28 ÷ 4 = 7**.
- For the set **{ 7, 11, 23, 5, 5 }**, the mean is $\frac{7+11+23+5+5}{5}$ = **51 ÷ 5 = $10\frac{1}{5}$**.

The median is the value that is exactly in the middle of the set when it is ordered.
If there are an even number of values, then the median is the mean of the two middle values.

- For the set **{ 5, 5, 7, 11 }**, the median is **(5 + 7) ÷ 2 = 6.**
- For the set **{ 7, 11, 23, 5, 5 }**, we begin by ordering the set of values: **{ 5, 5, 7, 11, 23 }**.

  The median is the middle value, **7.**

The mode is the value that occurs the greatest number of times.

- For the set **{ 5, 5, 7, 11 }**, the mode is **5.**

A set with every value listed an equal number of times is said to have no mode.

- For the set **{ 5, 5, 7, 7, 8, 8 }**, there is no mode.

Probability

The probability of an event is a value that expresses how likely an event is to occur.

- If the event is impossible, then the probability is **0**.
- If the event is certain, then the probability is **1**.
- All probabilities are between **0** and **1** inclusive.

The probability is found by dividing the number of times an event does occur, by the total number of times the event can possibly occur.

- The probability of rolling an odd number on a die is $\frac{3}{6}$ or $\frac{1}{2}$.

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