MATHS OLYMPIAD CONTEST PROBLEMS

VOLUME 5

CONTAINS APSMO MATHS OLYMPIAD PAPERS
FROM 2018 TO 2022

EXPLORING MATHS THROUGH PROBLEM SOLVING

Contents

About This Book

As a compendium of Maths Olympiad papers from 2018 to 2022, this book also reflects the growth and development of APSMO over that period of time.

The period was in fact one of significant change and progress.

Teachers who have been involved with the APSMO competitions and programs for a long time may recall that the first change appeared in 2018 when we revised our approach to “timing” in the Maths Olympiad contests. Historically, we were prescriptive about the length of time it should take students to complete each question in a contest. However, as the number of schools and students participating increased, so did the level of creativity displayed in problem solving and the variety of strategies that students were able to apply.

As a result, we identified the need to provide greater flexibility within the overall contest time and in 2018 we moved to a time limit of exactly 30 minutes for each Olympiad competition paper. We retained a suggested “time” for each question in that year, allowing a few minutes for individual student review, however by 2020 the time suggestions for individual questions were removed entirely.

2020 also saw education systems around the world grapple with the start of the COVID-19 pandemic. The beginning of the first lockdown period preceded the first APSMO contest of 2020 and resulted in the immediate reconfiguration of the Maths Olympiad to become, for one year only, an entirely “within-school” enrichment program.

As schools were unable to run the Olympiads in person under test conditions, we made the unprecedented decision to develop additional teaching resources and permit schools to use those resources and the 2020 contest papers as an intra-school program. Special Awards for participating students were created, to replace our usual competition awards, as we continued to assist teachers and maintain our commitment to developing students’ problem solving skills.

Since 2020 we have also invested further in the development of solutions for the Maths Olympiad. Modifications have been made to the way in which the solutions are presented – built on the solution format that had been developed for the APSMO Maths Games since 2016.

As illustrated in this book, we now provide longer form solutions, with more solution strategies. Prior to 2020, solutions for all five problems in a contest were arranged on a double A4 page. We now provide an entire page of solutions for each problem. This allows us to present more solution strategies, clearer diagrams, and expanded explanations to help both teachers and students alike.

We hope that you find the 2018-2022 sets of questions to be as interesting and thought-provoking as those from years past, and we look forward to many more years of mathematical challenges to come.

Problem Solving Strategies

Some of the most commonly used general strategies include:

Guess, Check and Refine.

This is the most basic of methods. Rarely is it quick or the one that the problem writer had in mind, but it can help you find the answer.

Even if your answer is right, you should ask other people, or check the “official solution” to understand the mathematics behind the problem. After all, the long-term goal is to improve mathematically, not merely to score points.

Draw a diagram.

One picture is worth a thousand words. In a geometry problem, always draw a diagram.

In a non-geometry problem, try to represent the amounts by a diagram; it often makes the solution much easier to understand.

Label it with all given measurements and see what other numbers you can figure out.

Find a pattern.

Mathematics is sometimes described as the science of patterns.

Studying how the numbers behave in a given problem can allow you to predict the result.

Make an organised list.

Listing every possibility in an organised way is an important tool.

How you organise your data often reveals additional information.

Build a table.

This is a special case of making an organised list.

Seeing the numbers laid out in an organised way allows you to comprehend the patterns within.

Solve a simpler related problem.

Many hard problems are merely easy ones that have been extended to larger numbers. Replacing the large number by at least three of the smallest possible numbers can introduce patterns that allow you to solve the original problem.

How to Use This Book

Establishing a Study Schedule

A little learning every day is more effective than large chunks of learning once a month, for two reasons. The mind needs time to absorb each new thought, and constant practice allows frequent review of previously learned concepts and skills. Together, these foster retention. Try to spend 10 or 15 minutes daily doing one or two problems. This approach should help you minimise the time needed to develop the ability to think mathematically.

You might want to track growth over time by recording success rates for these problem sets. Since you are probably changing the way you think about mathematics, your growth needs time to become apparent. Before long, you are likely to find solving problems intensely and increasingly rewarding.

Choosing Problems

What criteria can help you choose problems from this book? You might pick those that appeal to you, or those of a specific type, or you might go through all contests in order. You might want to select the easiest problems or the most difficult. Ultimately, you should test yourself against each and every problem. If you are stuck, try using the hint for the question.

Using the Solutions in This Book

Whether you solve a problem quickly or you are baffled, it is worth studying the solutions in this book, because often they offer unexpected insights that can help you understand the problem more fully. After you have invested time trying to solve each problem any way you can, reviewing our solutions is very effective. Think of each problem as a small doorway that opens into a large room of mathematical thought. In that room, you will find a wealth of concepts bundled together into one or more solutions to that problem, and you learn to think mathematically.

Many of the problems in this book can be solved in more than one way. There is always a single answer, but there can be many paths to that answer. If you don't see a mathematically efficient solution, try any method that promises to work. Once you solve a problem, go back and see if you can solve it by another method. See how many methods you can find. Then check our solutions to see if any of them differ from yours. Most veteran problem-solvers agree that the solutions usually teach much more than the questions do.

There is a saying: "All of us together are smarter than any of us alone." Attempting a problem as part of a study group enables you to benefit not only from your insights but also to learn from others. While you may want to test yourself as an individual, consider working with others on problems or, alternatively, comparing solutions aloud after each of you worked separately on the same questions.

Set 3: Olympiad 1

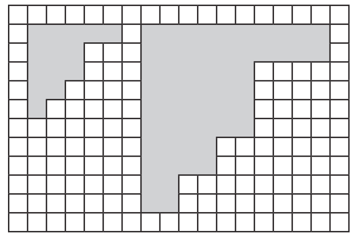
1A. Evaluate:

72 + 63 + 54 + 45 + 36 + 27.

1B. The smaller figure in the diagram has an area of 14 square units.

The larger figure on the right is produced by doubling the side lengths of the smaller figure.

What is the area of the larger figure, in square units?



1C. In a survey, 80 students were asked if they liked vanilla or chocolate cake.

Their responses were:

- 35 students liked vanilla,
- 32 students liked chocolate, and
- 24 students said they didn't like either vanilla or chocolate.

How many students said they liked both vanilla and chocolate?

1D. An *addy* number is a five-digit number with the following properties:

- The first (leftmost) digit plus the second digit is the third digit.
- The second digit plus the third digit is the fourth digit.
- The third digit plus the fourth digit is the fifth (rightmost) digit.
- All of the digits are different.

How many different *addy* numbers are possible?

1E. Sophia has three stickers for every two stickers that Ethan has.

If Sophia gives Ethan 16 of her stickers, then Ethan will have twice as many stickers as Sophia has.

How many stickers do Sophia and Ethan have all together?

Set 3: Hints

Set 3: Olympiad 1

1A. Rewrite the question to make it easier to solve.

1B. Investigate what happens to the area of a square when its dimensions are doubled.

1C. A Venn diagram may help you answer the question.

1D. Guess, Check and Refine your answer to find the 5 digit number.

1E. Given Sophie has more than 16 stickers and Ethan has 2 stickers for every 3 stickers Sophie has, how many could they both have had at the start?

Set 3: Olympiad 2

2A. How many 28s will you have once all the 28s are combined? How many 10s will you have once all the 10s are combined?

2B. Can you change Aidan's cards to get a number which increases by 4?

2C. What are the factors of 2020? How could you use these factors to determine the rectangle with the greatest perimeter?

2D. Write an organised list.

2E. Drawing a table could help you find the different outcomes.

Set 3: Olympiad 3

3A. How could you regroup the numbers to make this question easier to solve?

3B. Can you rearrange the digits to get a bigger number? Are there any other possible solutions?

3C. You could find the areas of squares with consecutive dimensions.

3D. Use a systematic method for finding the number of paths you could take.

3E. To keep track of the number of blocks with at least 2 faces showing you could number the blocks on your diagram as you count.

Set 3 Olympiad 1 Solutions

1A. **METHOD 1** ***Strategy:*** *Group the first and last, the second and the next-to-last, etc.*

$$\begin{aligned} 72 + 63 + 54 + 45 + 36 + 27 &= (72 + 27) + (63 + 36) + (54 + 45) \\ &= 99 + 99 + 99 \\ &= 3 \times (100 - 1) \\ &= 300 - 3 \\ &= \mathbf{297}. \end{aligned}$$

METHOD 2 ***Strategy:*** *Consider the tens digits and the ones digits separately.*

$$\begin{aligned} 72 + 63 + 54 + 45 + 36 + 27 &= (70 + 60 + 50 + 40 + 30 + 20) + (2 + 3 + 4 + 5 + 6 + 7) \\ &= 270 + 27 \\ &= \mathbf{297}. \end{aligned}$$

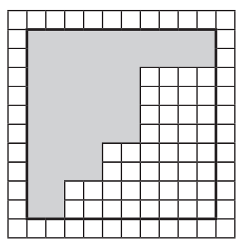
***Follow-Up**: (1) Form a 3 × 3 magic square using the numbers 18, 27, 36, 45, 54, 63, 72, 81, and 90.*

63	72	27
18	54	90
81	36	45

(2) The numbers in the following sequence keep doubling: 1, 2, 4, 8, and so on. If the numbers are added in the order 1 + 2 + 4 + 8 + ..., how many numbers are needed in order to first surpass one million? [20]

1B. **METHOD 1** ***Strategy:*** *Use the two long sides of the figure to draw a rectangle.*

Create a rectangle and subtract the area that is not in the original figure.



The dimensions of the rectangle will be **10** units × **10** units, so it will have an area of **100** square units.

Then apply some method to determine the area that is not in the original figure - for example, by counting the number of **1 × 1** squares, or subdividing the unwanted portion into rectangles.

In either case, the unwanted region has an area of **44** square units.

The area of the figure on the right is **100 – 44 = 56** square units.

Set 3 Olympiad 1 Solutions

1B. **METHOD 2** ***Strategy:*** *Use the idea that doubling the length dimensions of a shape multiplies the area by 4.*

Multiply the area of the smaller figure on the left by **4** to find the area of the larger figure.

The larger figure has an area of **4 × 14 = 56** units².

Follow-Up: *(1) Find the area of a shape formed by halving each of the sides of the shape on the left.* [3.5 units²]

(2) If the shape on the left is uniformly enlarged to create a similar shape that has an area of 224 square units, what is the perimeter of the resulting shape? [80 units]

1C. **METHOD 1** ***Strategy:*** *Apply logic and reasoning.*

There are 80 students.
24 students do not like either flavour.

						x	x	x	x
x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x

35 students like vanilla.

v	v	v	v	v	v	v	v	v	v
v	v	v	v	v	v	v	v	v	v
v	v	v	v	v	v	v	v	v	v
v	v	v	v	v					
						x	x	x	x
x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x

The remaining 80 – 24 – 35 = 21 students must only like chocolate.

v	v	v	v	v	v	v	v	v	v
v	v	v	v	v	v	v	v	v	v
v	v	v	v	v	v	v	v	v	v
v	v	v	v	v	c	c	c	c	c
c	c	c	c	c	c	c	c	c	c
c	c	c	c	c	c	x	x	x	x
x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x

Since 32 students like chocolate,
32 – 21 = **11** students must like both.

v	v	v	v	v	v	v	v	v	v
v	v	v	v	v	v	v	v	v	v
v	v	v	v	v c	v c	v c	v c	v c	v c
v c	v c	v c	v c	v c	c	c	c	c	c
c	c	c	c	c	c	c	c	c	c
c	c	c	c	c	c	x	x	x	x
x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x

Set 3 Olympiad 1 Solutions

1C. **METHOD 2** ***Strategy:*** *Draw a Venn diagram and apply algebra.*

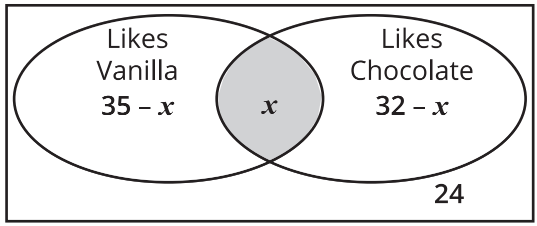
Let x indicate the number of students who like both vanilla and chocolate.

35 students like vanilla, so **35** – x only like vanilla (and not chocolate).

32 students like chocolate, so **32** – x only like chocolate (and not vanilla).

With **80** students in total,

$$(35 - x) + x + (32 - x) + 24 = 80$$
$$35 - x + x + 32 - x + 24 = 80$$
$$91 - x = 80,$$
$$x = 11.$$



There are **11** students who like both vanilla and chocolate.

Follow-Up*: How many integers between 1 and 150 inclusive are NOT divisible by 2, 3, or 5?* [40]

1D. **METHOD 1** ***Strategy:*** *Make an organised list.*

Each digit after the first two digits is the sum of the two previous digits.

Since no digit can be repeated, **0** cannot be either the first or second digit since the sum will repeat a digit.

1st	2nd	3rd	4th	5th	*addy* number
1	2	3	5	8	**12358**
1	3	4	7	11	This does not work, as **11** cannot be a digit. So no other number beginning with **1** will work.
2	1	3	4	7	**21347**
2	3	5	8	13	This does not work, as **13** cannot be a digit.
3	1	4	5	9	**31459**

No other numbers satisfy the requirements, so there are **3** *addy* numbers.

Set 3 Olympiad 1 Solutions

1D. **METHOD 2** ***Strategy:*** *Apply algebra.*

Let the number be *ABCDE*. Then:

$A + B = C$ —①

$B + C = D$ —②

$C + D = E$ —③

Substitute ① into ②:

$B + (A + B) = D$

$A + 2B = D$. —④

Substitute ① and ④ into ③:

$(A + B) + (A + 2B) = E$

$2A + 3B = E$.

Since E is a single digit, $2A + 3B \le 9$.

The only possible non-zero values for A and B are $(A, B) = (1, 2)$, $(2, 1)$, and $(3, 1)$.

Follow-Up: *How many 3-digit positive numbers have the units digit as the product of the tens digit and the hundreds digit if:*

(1) all three digits are distinct? [4: 236, 326, 248, 428]

(2) no digit is zero and at least two digits are the same?

[19: 111, 122, ..., 199, 212, 313, ..., 919, 224, 339]

(3) at least two of the digits are the same?

[28: as for part (2), plus 100, 200, ..., 900]

1E. **METHOD 1** ***Strategy:*** *Create a table to determine a pattern.*

Before giving any stickers to Ethan, Sophia must have a number of stickers that is:

- a multiple of **3**, and
- greater than **16**.

Start the table with **18** stickers for Sophia.

Before		After		Final Ratio	Verdict
Sophia	Ethan	Sophia	Ethan		
18	12	2	28	1:14	no good
21	14	5	30	1:6	closer
...	...	...	...	...	...
36	24	20	40	1:2	bingo!

Therefore, Sophia and Ethan started with **36 + 24 = 60** stickers all together.

METHOD 2 ***Strategy:*** *Apply algebra.*

Suppose Sophia began with $3N$ stickers.

Ethan began with $2N$ stickers.

All together, they had $5N$ stickers.

After the transfer, Ethan has $2N + 16$, Sophia has $3N - 16$.

Ethan now has twice as many as Sophia.

So: $2 \times (3N - 16) = 2N + 16$

$6N - 32 = 2N + 16$

$4N = 48$

$N = 12$.

Therefore, Sophia and Ethan began with **5 × 12 = 60** stickers.

Follow-Up: *Emma has three stickers for every four stickers that Steven has. Together they have more than 125 stickers, but less than 130 stickers. How many stickers must Steven give Emma so that they each have the same number of stickers?* [9]

Problem Types

Many, but not all, contest problems can be categorised. This is useful if you choose to work with several related problems, even if they involve different concepts.

Problems are organised by type and are coded by page number and problem placement on that page. For example, "Clock Problems: 29A, 37B, 182C" refers to three questions, each involving clocks: problem A on page 29, problem B on page 37, and problem C on page 182.

Contest problems from the different divisions are arranged as follows:

- **Division J:** pages 14 – 38
- **Division S:** pages 160 – 184

Each Follow-Up problem is located after a model solution to a contest problem, is related conceptually to it, and usually extends or expands an aspect of it.

Problem Types

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