2023 Maths Olympiads Division S Preparation Kit

Welcome to the APSMO Maths Olympiads for 2023.

This year, instead of scheduling five competition papers throughout the school year, we are replacing the first competition paper with this Preparation Kit, so that the format of the Maths Olympiads for 2023 will be:

- Preparation Kit: Available from February, 2023
- Competition 1: Wednesday May 3, 2023
- Competition 2: Wednesday June 14, 2023
- Competition 3: Wednesday July 26, 2023
- Competition 4: Wednesday September 6, 2023

Preparing for the APSMO Maths Olympiads

The purpose of this Preparation Kit is to provide students with an opportunity to familiarise themselves with the concepts, and terminology, that will subsequently be used in the four competition papers for 2023.

For each of the problems in this kit, a number of different solution methods are suggested, so that students can be exposed to multiple ways of approaching mathematical problems.

The kit additionally includes an updated reference sheet for relevant skills and terminology. This reference sheet can also be found in the Resources section of your Members Portal.

Examples of how this kit may be used include:

- Reinforcing previously learned concepts and terminology
- Introducing new or different solution methods
- Providing diagrams that support a teacher's or student's explanations
- Offering problem-solving homework
- Supporting students' own study as a standalone resource

Further questions and solution methods can also be found in the APSMO resource books, available from www.apsmo.edu.au.

MATHS OLYMPIAD

A. 2^n means that 2 is multiplied by itself n times.
For example, 2^2 means $2 \times 2 = 4$.
Since the ones digit of 4 is 4, we say that 2^2 has 4 in the ones place.
Likewise, 2^4 means $2 \times 2 \times 2 \times 2 = 16$.
The ones digit of 16 is 6, so 2^4 has 6 in the ones place.
2^{20} is 2 multiplied by itself 20 times.
What is the ones digit in the value of 2^{20}?

B. *A* is a number in a data set of five numbers:

1.8 1.6 2.1 1.7 *A*

The mean, the median, and the mode of the five numbers are all equal.

What number does *A* represent?

C. The school band has woodwind, brass, and percussion instruments only.

20 students play woodwind instruments.

28 students play brass instruments.

One fifth of the students play percussion instruments.

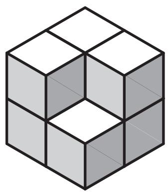
How many students are there in the school band?

D. Jason has eight small wooden cubes, each with an edge length of 1 cm.

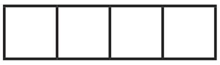
He arranges them to make a larger cube with an edge length of 2 cm.

Then he removes one of the small cubes from a corner of the larger cube.

What is the surface area, in square centimetres, of the remaining object (including its bottom surface)?



E. Four digits are arranged in a row to form a 4-digit number.



What is the largest possible 4-digit number, where the product of the digits is 56?

Write your answers in the boxes on the back.

← *Keep your answers hidden by folding backwards on this line.*

MATHS OLYMPIAD

A.	*Fold here. Keep your answers hidden.*
B.	
C.	
D.	
E.	

Write your answers in the boxes on the back.

← *Keep your answers hidden by folding backwards on this line.*

F. A cryptarithm is a mathematical puzzle where digits in a calculation have been replaced by letters.

$$\begin{array}{r} C\ A\ T \\ +\ C\ A\ T \\ \hline M\ E\ O\ W \end{array}$$

Different letters represent different digits.

What is the greatest possible value represented by *MEOW*?

G. A prime number is a number that is only divisible by 1 and itself.

The length and width of a rectangle are both a prime number of centimetres.

The rectangle has a perimeter of 30 centimetres.

What is the area of the rectangle, in square centimetres?

H. What is the probability that a randomly selected three-digit positive integer has no repeated digits?

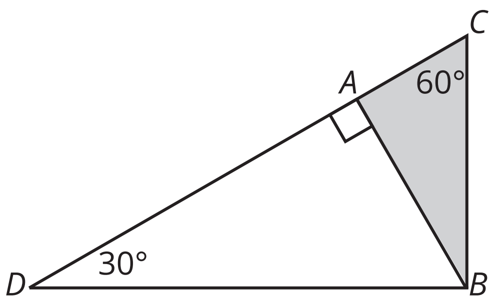
Express your answer both as a fraction in lowest terms, and in decimal form.

I. $\triangle ABC$ and $\triangle BDC$ both have angles of 30°, 60° and 90°.

They both share side *BC*.

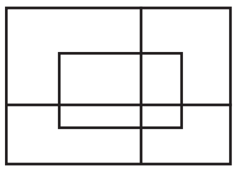
$\triangle ABC$ is shaded and is contained within $\triangle BDC$.

What percentage of the total area of $\triangle BDC$ is shaded?



J. In the diagram, all angles are right angles.

How many different rectangles can be traced along the lines in this diagram?



F.	*Fold here. Keep your answers hidden.*
G.	
H.	
I.	
J.	

Example Solution A

2^n means that 2 is multiplied by itself n times. For example, 2^2 means $2 \times 2 = 4$. Since the ones digit of 4 is 4, we can say that 2^2 has 4 in the ones place.

Likewise, 2^4 means $2 \times 2 \times 2 \times 2 = 16$. The ones digit of 16 is 6, so 2^4 has 6 in the ones place.

2^{20} is 2 multiplied by itself 20 times. What is the ones digit in the value of 2^{20}?

Strategy: Solve a Simpler Related Problem

Multiplying out 2^{20} looks very long and complicated. However, we only need to find the ones digit.

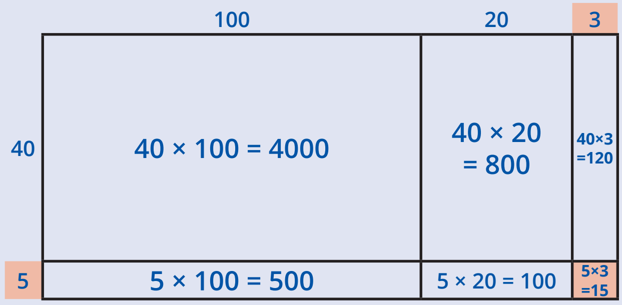
Can we solve this problem without actually finding the value of 2^{20}?

Let's consider what happens when we multiply large numbers - say, 123×45.

Whether we use a written algorithm, or the area model, we can see that:

- the only part of 123 that affects the ones digit of the result is the 3, and
- the only part of 45 that affects the ones digit of the result is the 5.

	1	2	3
×		4	5
	6	1	5
4	9	2	0
5	5	3	5



The other place values represent multiples of 10, and so they will have no effect on the ones digit of the result.

Method 1: Multiply out 2^{20}

Since $2^4 = 2 \times 2 \times 2 \times 2 = 16$,
then $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 16 \times 2 = 32$.

So the ones digit of 2^5 is 2.

Note that, instead of $16 \times 2 = 32$, we could have worked out $6 \times 2 = 12$.

The ones digit in the result will be the same.

$$2^{10} = \underbrace{2\times2\times2\times2\times2\times2\times2\times2\times2\times2}_{\text{10 times}}$$
$$= \underbrace{2\times2\times2\times2\times2}_{\text{5 times}} \times \underbrace{2\times2\times2\times2\times2}_{\text{5 times}}$$
$$= 2^5 \times 2^5$$
$$= 32 \times 32$$
$$= 1024.$$

So 2^{10} has 4 in the ones place.

Again, because 2^5 has 2 in the ones place, we could just have worked out $2 \times 2 = 4$.

$$2^{20} = \underbrace{2 \times 2 \times \ldots \times \ldots \times 2 \times 2}_{\text{20 times}}$$
$$= \underbrace{2 \times 2 \times \ldots \times 2}_{\text{10 times}} \times \underbrace{2 \times \ldots \times 2}_{\text{10 times}}$$
$$= 2^{10} \times 2^{10}$$
$$= 1024 \times 1024.$$

To find the ones digit of the product, we don't need to multiply it all out.

Since $2^{20} = 2^{10} \times 2^{10}$, the ones digit for 2^{20} is the same as the ones digit for $4 \times 4 = 16$.

Therefore the ones digit of 2^{20} is 6.

Method 2: Find a Pattern

When we list the values for 2^1, 2^2, 2^3, 2^4 and so on, the ones digits appear to occur in the order 2, 4, 8, 6, repeat.

With a four-digit pattern, every fourth position will have the same digit.

Therefore the ones digit for 2^4, 2^8, 2^{12}, and every 2^n where n is a multiple of 4, would have the same digit.

Since 20 is a multiple of 4, from the pattern we can reason that the ones digit of 2^{20} will also be 6.

	Value	Ones digit
$2^1 = 2$	2	2
$2^2 = 2 \times 2$	4	4
$2^3 = 2 \times 2 \times 2$	8	8
$2^4 = 2 \times 2 \times 2 \times 2$	16	6
$2^5 = 2 \times 2 \times 2 \times 2 \times 2$	32	2
$2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$	64	4
$2^7 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$	128	8
$2^8 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$	256	6
$2^9 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$	512	2
$2^{10} = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$	1024	4
$2^{11} = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$	2048	8
$2^{12} = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$	4096	6

Answer: 6

Example Solution B

A is a number in a data set of five numbers:

1.8 1.6 2.1 1.7 *A*

The mean, the median, and the mode of the five numbers are all equal.

What number does *A* represent?

Mean

The mean of the five values can be found by:

- adding the five values together, and
- dividing the sum by the number of values (that is, dividing the sum by five).

Median

The median of the five values can be found by:

- rearranging them in ascending order, and
- identifying the middle value in the sequence.

Mode

The mode of the five values is the value that occurs the most times within the set.

Strategy: Eliminate All But One Possibility

We want to find a value for ***A*** so that the mean, the median, and the mode are all equal.

Mode

If we do not include the value ***A***, then there is no mode in the remaining set of four numbers:

1.8 1.6 2.1 1.7

For a mode to exist, ***A*** must be equal to one of these four numbers.

It may here be worth noting that, irrespective of which value is taken by ***A***, the only possible values for the median are the two middle values (**1.7** and **1.8**).

Value for *A*	Median	Mean	Summary
1.6	1.6 1.6 (1.7) 1.8 2.1	1.6 + 1.7 + 1.8 + 2.1 = 7.2 $\frac{7.2 + 1.6}{5} = \frac{8.8}{5} = 1.76$	Mean: 1.76 Median: 1.7 Mode: 1.6
1.7	1.6 1.7 (1.7) 1.8 2.1	$\frac{7.2 + 1.7}{5} = \frac{8.9}{5} = 1.78$	Mean: 1.78 Median: 1.7 Mode: 1.7
1.8	1.6 1.7 (1.8) 1.8 2.1	$\frac{7.2 + 1.8}{5} = \frac{9.0}{5} = 1.8$	Mean: 1.8 Median: 1.8 Mode: 1.8
2.1	1.6 1.7 (1.8) 2.1 2.1	$\frac{7.2 + 2.1}{5} = \frac{9.3}{5} = 1.86$	Mean: 1.86 Median: 1.8 Mode: 2.1

The mean, the median and the mode are all equal to **1.8** when ***A*** represents **1.8**.

Answer: 1.8

Example Solution C

The school band has woodwind, brass, and percussion instruments only.

20 students play woodwind instruments.

28 students play brass instruments.

One fifth of the students play percussion instruments.

How many students are there in the school band?

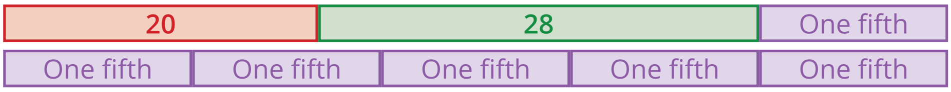
Strategy 1: Work Backwards

In the school band, 20 students play woodwind instruments.

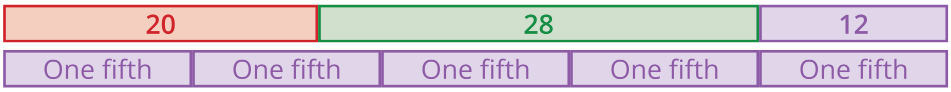
28 students play brass instruments.

One-fifth of the students play percussion instruments.

If one-fifth of the students play percussion instruments, then four-fifths of the students must play woodwind or brass.



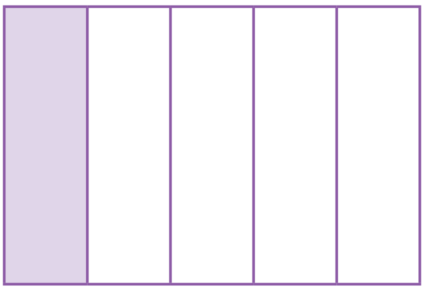
Four-fifths of the students is equivalent to 20 + 28 = 48 students, so one-fifth must be 48 ÷ 4 = 12 students.



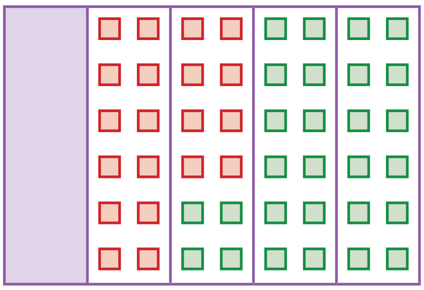
There are 20 + 28 + 12 = **60** students in the band.

Strategy 2: Work Backwards (Alternative Method)

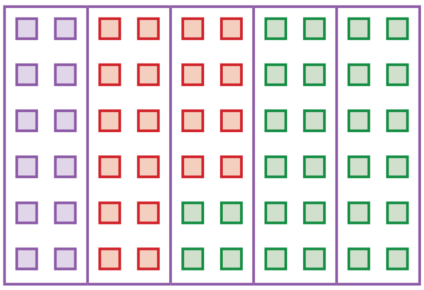
One-fifth of the students play percussion instruments.



With 20 woodwind and 28 brass, there are 20 + 28 = 48 other students in the other four-fifths of the band.



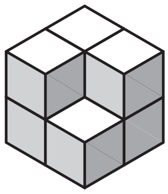
With **48** students in four-fifths of the band, one-fifth of the band comprises **48** ÷ 4 = 12 students.



There are 5 × 12 = **60** students in the band.

Answer: 60

Example Solution D



Jason has eight small wooden cubes, each with an edge length of 1 cm.

He arranges them to make a larger cube with an edge length of 2 cm.

Then he removes one of the small cubes from a corner of the larger cube.

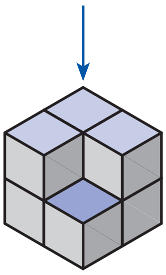
What is the surface area, in square centimetres, of the remaining object (including its bottom surface)?

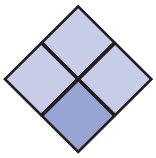
Strategy 1: Divide a Complex Shape (1)

Let's consider how the object looks from each direction.

Looking straight down at the top of the cube, we would see three squares on the top level and one square that is a bit further away.

That's 4 square centimetres of surface area.

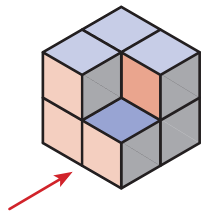


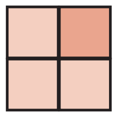


Top View

Looking at the cube from the left side, we would see three squares on the near face and one square that is a bit further away.

Again, that's 4 square centimetres of surface area.

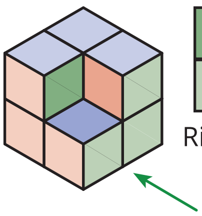


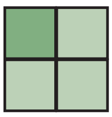


Left View

Looking at the cube from the right side, we would see three squares on the near face and one square that is a bit further away.

Once again, that's 4 square centimetres of surface area.



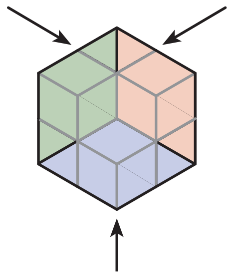


Right View

A cube has six faces.

We know that the 3 faces that we cannot see were not affected by the removal of the 1 cm^3 cube.

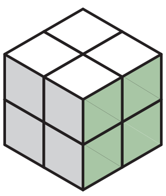
So those faces will each have 4 square centimetres of surface area.



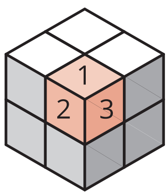
Therefore the total surface area is **6 × 4 = 24** square centimetres.

Strategy 2: Divide a Complex Shape (2)

Jason's original object was made up of eight cubes.

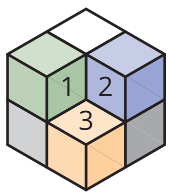


Each of the eight cubes contributes three faces to the surface area of the object.



When Jason removed one cube, he:

- removed the three faces that had been contributed by that cube, and
- also exposed three faces that had previously been hidden by that cube.



The 8 cm^3 cube had six 4 cm^2 faces, for a total surface area of 6 × 4 = 24 cm^2.

Alternatively, with 8 cubes in the original object, each contributing three 1 cm^2 faces to the surface area, the original object had a total surface area of 8 × 3 cm^2 = 24 cm^2.

After removing one of the 1 cm^3 cubes in the object, the total surface area remains the same.

Therefore the exposed surface area of the object is **24** cm^2.

Answer: 24 cm^2

Example Solution E

Four digits are arranged in a row to form a 4-digit number.

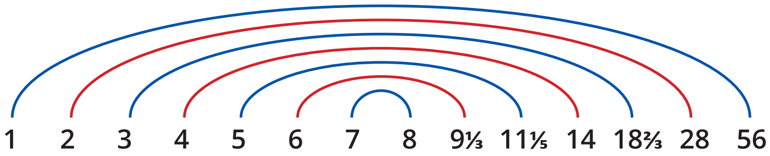
What is the largest possible 4-digit number, where the product of the digits is 56?

Strategy: Make an Organised List, and Eliminate All But One Possibility

Since the product of the digits is **56**, we might begin by finding all of the factors of **56**.

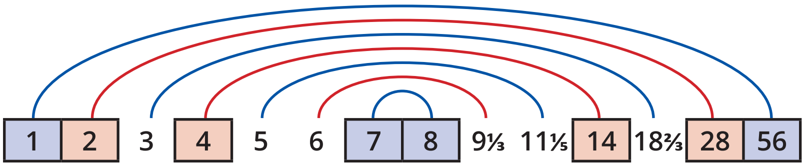
Method 1: Find factor pairs

We can find all of the possible factors by systematically checking every integer, starting from **1**.



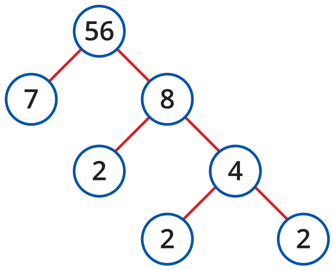
Working our way up from **1**, we can see that **1**, **2**, **4** and **7** will divide evenly into **56**.

The other factor in each factor pair, **56**, **28**, **14** and **8**, will likewise divide evenly into **56**.



Since we have established divisibility for every value up to **7**, we do not need to check beyond **7** × **8**.

Method 2: Use a tree diagram



The prime factors of **56** are 2 × 2 × 2 × 7.

We can combine the prime factors as follows, to generate the complete list of factors.

$2^0 \times 7^0 = 1$	$2^0 \times 7^1 = 7$
$2^1 \times 7^0 = 2$	$2^1 \times 7^1 = 14$
$2^2 \times 7^0 = 4$	$2^2 \times 7^1 = 28$
$2^3 \times 7^0 = 8$	$2^3 \times 7^1 = 56$

The factors of **56** are **1**, **2**, **4**, **7**, **8**, **14**, **28** and **56**.

The greatest single-digit factor of **56** is 8, so we will place 8 in the thousands place.

8			

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56 ÷ 8 = 7, so the product of the remaining digits must be 7.

The greatest single-digit factor of 7 is 7, so we will place 7 in the hundreds place.

8	7		

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7 ÷ 7 = 1, so the product of the remaining digits must be 1.

The last two digits must therefore both be 1.

8	7	1	1

The largest possible 4-digit number, where the product of the digits is 56, is **8711**.

Answer: 8711

Example Solution F

A cryptarithm is a mathematical puzzle where digits in a calculation have been replaced by letters.

Different letters represent different digits.

What is the greatest possible value represented by *MEOW*?

$$\begin{array}{cccc} & C & A & T \\ + & C & A & T \\ \hline M & E & O & W \end{array}$$

Strategy: Eliminate All But One Possibility

To find the greatest possible value for ***MEOW***, we might begin by considering the greatest possible value for ***M***.

Suppose ***M*** represented 9.
Then, the greatest possible value for ***C*** would be 8.

$$\begin{array}{cccc} & 8 & A & T \\ + & 8 & A & T \\ \hline 9 & E & O & W \end{array}$$

Clearly, having ***C*** = 8 makes it impossible for ***M*** to be 9.

The greatest possible value for ***M*** is 1.

$$\begin{array}{cccc} & 8 & A & T \\ + & 8 & A & T \\ \hline 1 & 7 & O & W \end{array}$$

Using the preceding argument, we can see that the greatest possible value for ***MEOW*** requires us to maximise the value for ***C***.

Let's set ***C*** to 9.

Since 9 is taken, the greatest possible value for ***E*** would then be 8.

$$\begin{array}{cccc} & 9 & A & T \\ + & 9 & A & T \\ \hline 1 & 8 & O & W \end{array}$$

Since 9 and 8 are taken, the greatest possible value for ***O*** would be 7.

$$\begin{array}{cccc} & 9 & A & T \\ + & 9 & A & T \\ \hline 1 & 8 & 7 & W \end{array}$$

It's not possible for ***A*** + ***A*** = 7, but with trading from the ones place, we can have ***A*** + ***A*** + 1 = 7.

$$\begin{array}{cccc} & & 1 & \\ & 9 & 3 & T \\ + & 9 & 3 & T \\ \hline 1 & 8 & 7 & W \end{array}$$

9, 8, and 7 are now taken.

The greatest possible value for ***W*** is 6.

$$\begin{array}{cccc} & & 1 & \\ & 9 & 3 & T \\ + & 9 & 3 & T \\ \hline 1 & 8 & 7 & 6 \end{array}$$

If ***W*** is 6, then with the trading requirement we will have ***T*** + ***T*** = 16.

That cannot be right, because ***E*** = 8. So ***T*** cannot be 8.

$$\begin{array}{cccc} & & 1 & \\ & 9 & 3 & (8) \\ + & 9 & 3 & (8) \\ \hline 1 & \boxed{8} & 7 & 6 \end{array}$$

The next greatest possible value for ***W*** is 5. If so, we will have ***T*** + ***T*** = 15.

This is not possible, since ***T*** cannot be 7.5.

$$\begin{array}{cccc} & & 1 & \\ & 9 & 3 & T \\ + & 9 & 3 & T \\ \hline 1 & 8 & 7 & 5 \end{array}$$

The following values are likewise not possible:

- ***T*** + ***T*** = 14. ***T*** cannot be 7.
- ***T*** + ***T*** = 13. ***T*** cannot be 6.5.

$$\begin{array}{cccc} & & 1 & \\ & 9 & 3 & T \\ + & 9 & 3 & T \\ \hline 1 & 8 & 7 & W \end{array}$$

The next greatest possible value for ***W*** is 2.

If so, we will have ***T*** + ***T*** = 12, and so ***T*** = 6.

$$\begin{array}{cccc} & & 1 & \\ & 9 & 3 & 6 \\ + & 9 & 3 & 6 \\ \hline 1 & 8 & 7 & 2 \end{array}$$

The greatest possible value for ***MEOW*** is **1872**.

Answer: 1872

Example Solution G

A prime number is a number that is only divisible by 1 and itself.

The length and width of a rectangle are both a prime number of centimetres.

The rectangle has a perimeter of 30 centimetres. What is the area of the rectangle, in square centimetres?

Strategy 1: Make an Organised List

Since a prime number is only divisible by **1** and itself, it can only be written as a product of two whole numbers if those whole numbers are **1** and itself.

We can use this idea to list all of the prime numbers.

The first prime number is 2, because its only factors are **1** and **2**. Any further multiples of 2 are not prime.

1	2	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~	11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~	21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~

We can see that the next prime number must be 3. We can now eliminate further multiples of 3.

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~	~~21~~	~~22~~	23	~~24~~	25	~~26~~	~~27~~	~~28~~	29	~~30~~

The next prime number must be 5.

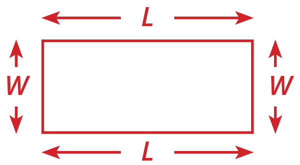
1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~	~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~

Continuing with this method (called Eratosthenes' Sieve), we would find that the prime numbers less than **30** are 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29.

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~	~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~

Let L be the length of the rectangle in cm, and W be the width of the rectangle in cm.

The perimeter of the rectangle is $L + W + L + W$, as shown.



The problem states that the perimeter of the rectangle is **30** cm.

$L + W + L + W = 30$ cm,

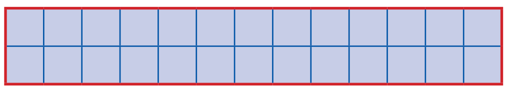
so $L + W = 30$ cm $\div$ 2

$= 15$ cm.

We can try different prime numbers for the L measurement, to see if the W is likewise prime.

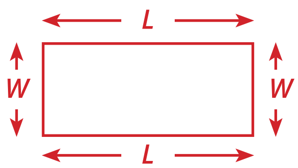
L	$W = 15 - L$	Both prime?
2	15 – 2 = 13	Yes
3	15 – 3 = 12	No
5	15 – 5 = 10	No
7	15 – 7 = 8	No
11	15 – 11 = 4	No
13	15 – 13 = 2	Yes

Since the rectangle is either 2 cm long and 13 cm wide, or 13 cm long and 2 cm wide, its area must be 13 cm × 2 cm = **26** cm^2.



Strategy 2: Consider Properties of Numbers

As noted in Strategy 1, the perimeter of a rectangle is equal to $L + W + L + W$.



Therefore,

$L + W + L + W = 30$ cm

and so $L + W = 15$ cm.

Since **15** is odd, L and W must have different parity (they cannot both be odd, or both even).

2 is the only prime number that is also even, so one of the side lengths must be 2 cm.

The other side length would then be 15 – 2 = 13 cm.

Therefore, the area of the rectangle must be 2 cm × 13 cm = **26** cm^2.

Answer: 26 cm^2

Example Solution H

What is the probability that a randomly selected three-digit positive integer has no repeated digits?
Express your answer both as a fraction in lowest terms, and in decimal form.

Strategy: Count in an Organised Way

We begin by finding the number of 3-digit positive integers.

With 9 possible values in the hundreds place,

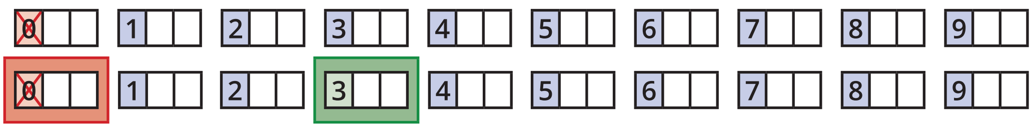
10 possible values in the tens place,

and 10 possible values in the ones place,

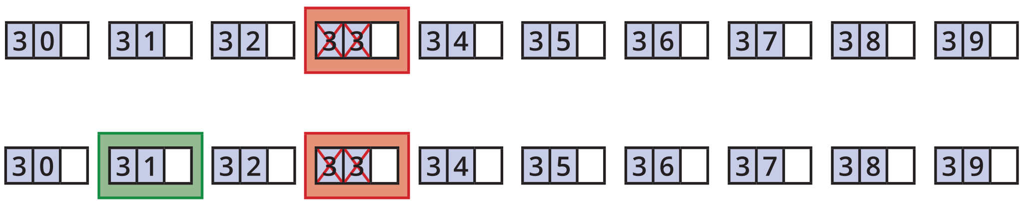
there are 9 × 10 × 10 = 900 different 3-digit positive integers.

To find how many of these 3-digit positive integers have no repeated digits:

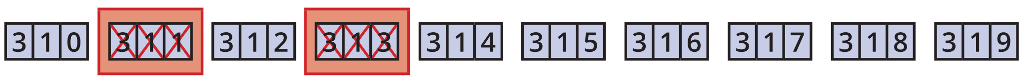
There are 9 possible values in the 100s place.
Let's suppose that the hundreds digit is 3.



If we want no repeated digits, there are now just 9 possible values in the tens place.
Let's suppose that the tens digit is 1.



If we want no repeated digits, there are now just 8 possible values in the ones place.



There are 9 × 9 × 8 = 648 3-digit positive integers that have no repeated digits.

The probability that a randomly selected 3-digit positive integer has no repeated digits, is $\frac{9 \times 9 \times 8}{9 \times 10 \times 10} = \frac{648}{900}$.

Fraction in Lowest Terms

$$\frac{9 \times 9 \times 8}{9 \times 10 \times 10} = \frac{9}{9} \times \frac{9 \times 2 \times 2 \times 2}{5 \times 2 \times 5 \times 2}$$
$$= 1 \times \frac{9 \times 2}{5 \times 5} \times \frac{2 \times 2}{2 \times 2}$$
$$= 1 \times \frac{18}{25} \times 1$$
$$= \frac{18}{25}$$

Decimal Form

$$\frac{9 \times 9 \times 8}{9 \times 10 \times 10} = \frac{9}{9} \times \frac{72}{100}$$
$$= 1 \times 0.72$$
$$= 0.72$$

As a fraction, the probability is $\frac{18}{25}$.

As a decimal, the probability is **0.72**.

Answer: $\frac{18}{25}$; 0.72

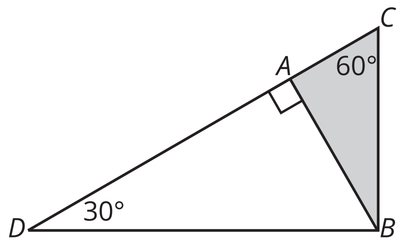
Example Solution I

△*ABC* and △*BDC* both have angles of 30^0, 60^0 and 90^0.

They both share side *BC*.

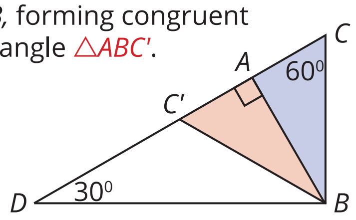
△*ABC* is shaded and is contained within △*BDC*.

What percentage of the total area of △*BDC* is shaded?

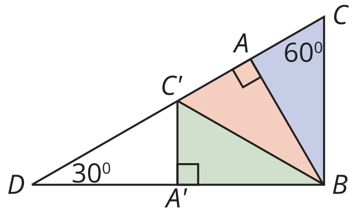


Strategy 1: Draw a Diagram (1)

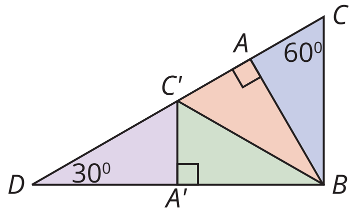
We begin by reflecting the smaller shaded triangle △*ABC* over side *AB*, forming congruent triangle △*ABC'*.



Next, we reflect △*ABC'* about *BC'* to create △*A'BC'*.



Notice that △*A'BC'* reflected over *A'C'* will coincide with △*A'DC'*.

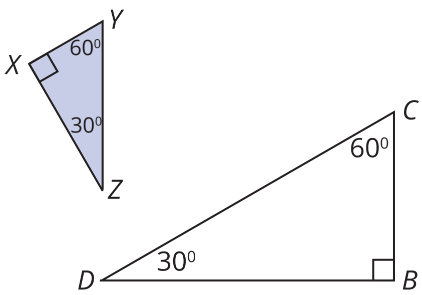


The three newly created triangles are all congruent (i.e. identical) to the original shaded triangle.

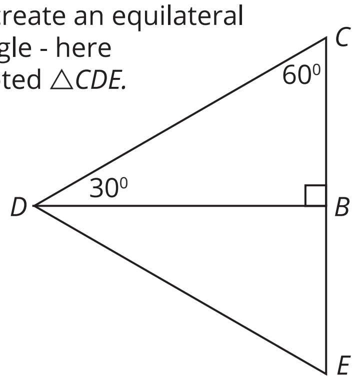
Therefore, the original shaded triangle is one quarter, or **25%**, of the area of △*BCD*.

Strategy 2: Draw a diagram (2)

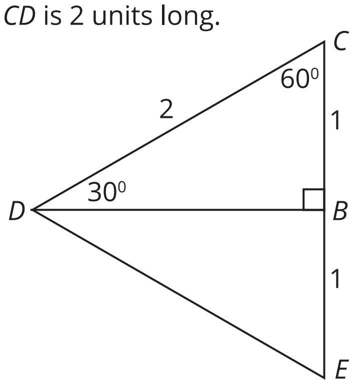
We begin by separating the two triangles.
We'll call the smaller one △*XYZ*.



∠*BCD* = 60^0 and ∠*CDB* = 30^0.
By reflecting △*BCD* over *BD*, we can create an equilateral triangle - here denoted △*CDE*.

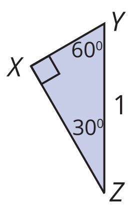


This means that *CE* = *CD*.
If we say that *CB* is 1 unit long, then *CD* is 2 units long.

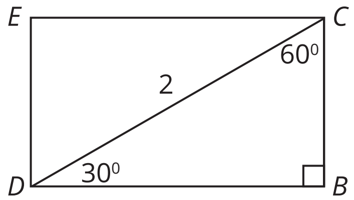


From our original diagram, we can see that *CB* is the same length as *YZ*.

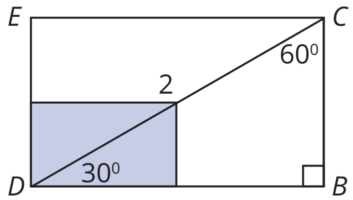
YZ must also be 1 unit long.



By repositioning the two larger triangles to form rectangle *BCED*, we can see that *YZ* (the long side of △*XYZ*) will be exactly half of the length of the diagonal.



A rectangle comprising two of these smaller triangles will have exactly one quarter of the area of the larger rectangle.



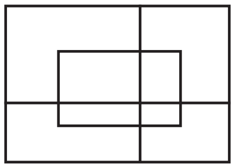
Therefore, the smaller triangle is exactly one quarter, or **25%**, of the area of the larger triangle.

Answer: 25%

Example Solution J

In the diagram, all angles are right angles.

How many different rectangles can be traced along the lines in this diagram?

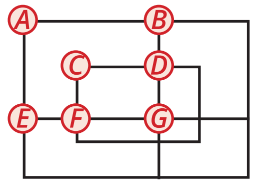


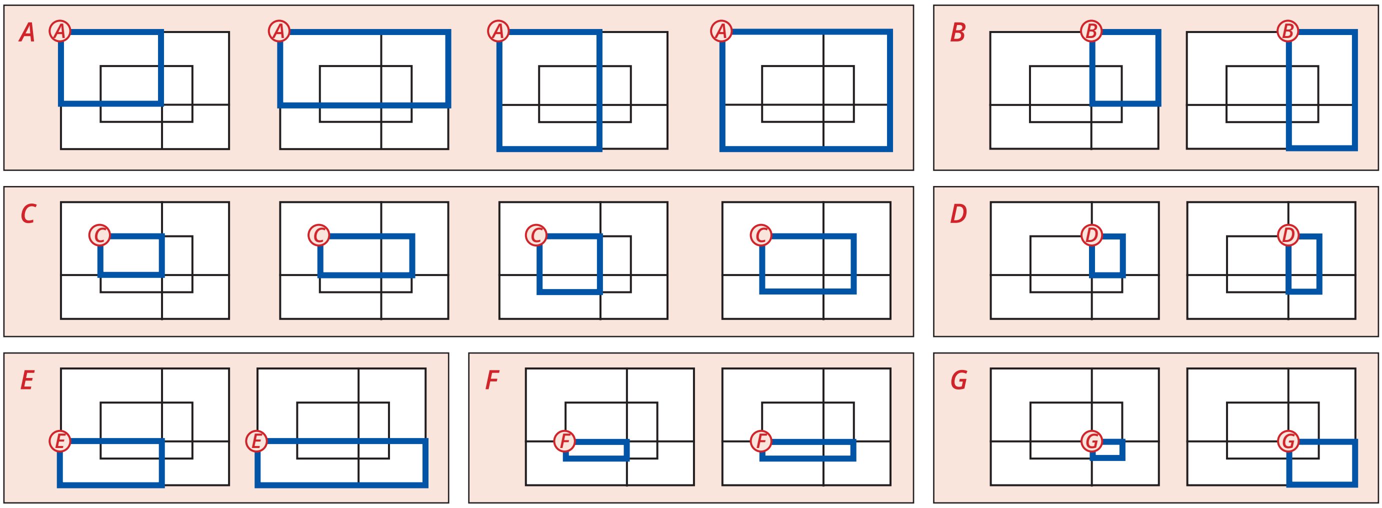
Strategy 1: Draw a Diagram

We could begin by marking all of the positions where we might have the top left corner of a rectangle.

Then, we can list all of the possible rectangles with top left corner *A*, *B*, and so on.

Working our way around systematically through all of the different positions, we can be sure that we have found them all.





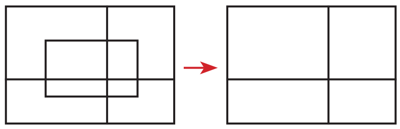
We can trace **18** different rectangles on the lines in this diagram.

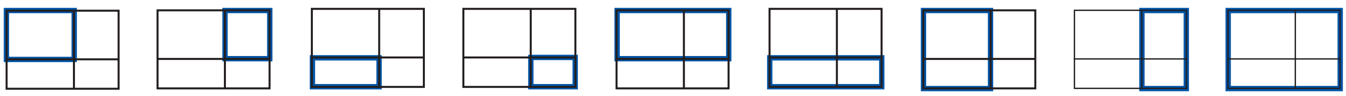
Strategy 2: Solve a Simpler Related Problem

We can make the problem easier to solve by first removing some of the lines in the diagram.

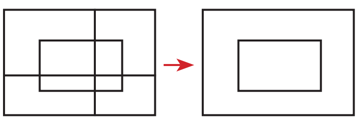
For example, we might begin by removing the inner rectangle.

The following **9** rectangles can be traced on the resulting diagram.

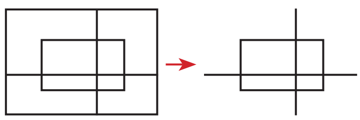


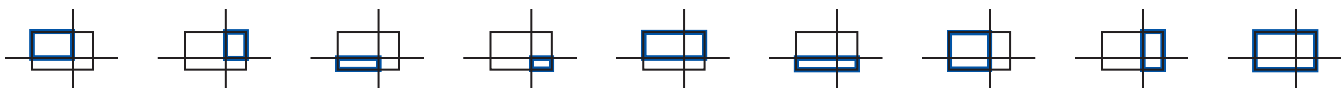


It is impossible for a rectangle to include sides from both the inner and outer rectangles.



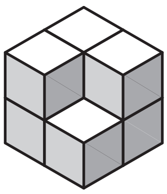
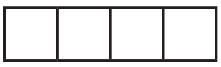
However, we can remove the outer rectangle, and trace another **9** rectangles on the resulting diagram.

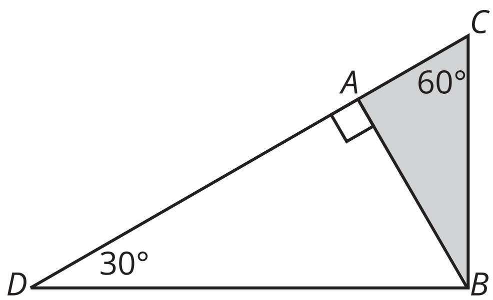
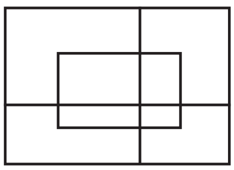




We can trace **9 + 9 = 18** different rectangles on the lines in this diagram.

Answer: 18

A. **Answer: 6**	2^n means that 2 is multiplied by itself *n* times. For example, 2^2 means 2 × 2 = 4. Since the ones digit of 4 is 4, we say that 2^2 has 4 in the ones place. Likewise, 2^4 means 2 × 2 × 2 × 2 = 16. The ones digit of 16 is 6, so 2^4 has 6 in the ones place. 2^{20} is 2 multiplied by itself 20 times. What is the ones digit in the value of 2^{20}?
B. **Answer: 1.8**	*A* is a number in a data set of five numbers: 1.8 1.6 2.1 1.7 *A* The mean, the median, and the mode of the five numbers are all equal. What number does *A* represent?
C. **Answer: 60**	The school band has woodwind, brass, and percussion instruments only. 20 students play woodwind instruments. 28 students play brass instruments. One fifth of the students play percussion instruments. How many students are there in the school band?
D. **Answer: 24** cm²	Jason has eight small wooden cubes, each with an edge length of 1 cm. He arranges them to make a larger cube with an edge length of 2 cm. Then he removes one of the small cubes from a corner of the larger cube. What is the surface area, in square centimetres, of the remaining object (including its bottom surface)? 
E. **Answer: 8711**	Four digits are arranged in a row to form a 4-digit number.  What is the largest possible 4-digit number, where the product of the digits is 56?

F. **Answer: 1872**	A cryptarithm is a mathematical puzzle where digits in a calculation have been replaced by letters. $\begin{array}{r} C\ A\ T \\ +\ C\ A\ T \\ \hline M\ E\ O\ W \end{array}$ Different letters represent different digits. What is the greatest possible value represented by *MEOW*?
G. **Answer: 26** cm^2	A prime number is a number that is only divisible by 1 and itself. The length and width of a rectangle are both a prime number of centimetres. The rectangle has a perimeter of 30 centimetres. What is the area of the rectangle, in square centimetres?
H. **Answer:** $\frac{18}{25}$**; 0.72**	What is the probability that a randomly selected three-digit positive integer has no repeated digits? Express your answer both as a fraction in lowest terms, and in decimal form.
I. **Answer: 25%**	$\triangle ABC$ and $\triangle BDC$ both have angles of 30°, 60° and 90°. They both share side *BC*. $\triangle ABC$ is shaded and is contained within $\triangle BDC$. What percentage of the total area of $\triangle BDC$ is shaded? 
J. **Answer: 18**	In the diagram, all angles are right angles. How many different rectangles can be traced along the lines in this diagram? 

Skills

Presenting Answers

Unless otherwise specified in a problem, **equivalent numbers or expressions are acceptable**.

- For example, $3\frac{1}{2}$, $\frac{7}{2}$, and 3.5 are equivalent.
- However, $3\frac{2}{4}$ and $\frac{70}{20}$ would not be considered to be correct, as they are not in lowest terms.

After reading a problem, it is useful to indicate the nature of the answer, before commencing the solution strategy.

For example:

- **“*A* = ____ , *B* = ____ .”**
- **“The largest number is ____ .”**
- **"The [sum | difference | product | quotient] is ____ ."**
- **"The probability, as a [fraction | decimal | percentage], is ____ ."**
- **"The perimeter is ____ centimetres."**
- **"The area is ____ square units."**
- **“The average speed is ____ kilometres per hour.”**

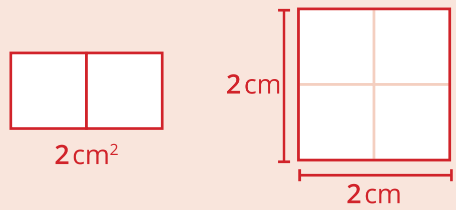
Units of Measurement

Familiarity with units of measurement is assumed, including conversions from one unit to another:

- **Time**: seconds ↔ minutes ↔ hours ↔ days
- **Length**: millimetres ↔ centimetres ↔ metres ↔ kilometres
- **Area**: mm^2 ↔ cm^2 ↔ m^2 ↔ km^2
- **Volume**: mm^3 ↔ cm^3 ↔ m^3; 1 cm^3 = 1 millilitre; mL ↔ litres
- **Mass**: grams ↔ kilograms
- **Angles**: degrees (°)
- **Temperature**: degrees Celsius (°C)

Units of measurement must be correct if given in an answer.

To avoid confusion, read cm^2 as "square centimetres", not "centimetres squared".



2 square centimetres | Possible misinterpretation

Order of Operations

When an expression has more than one arithmetic symbol, certain operations occur before others.

There are a few ways to remember the order of operations, and mnemonics are often used (e.g. BIDMAS; PEMDAS).

However, it can also be useful to consider the intent when an arithmetic expression is constructed.

By convention, we observe the following priorities:

1. Perform operations in **parentheses**, **braces**, or **brackets**. The **vinculum** (line in a fraction) is also considered as a grouping symbol, similar to parentheses.
2. Evaluate **exponents** (**indices**).
3. Evaluate **multiplication** and **division**, from left to right.
4. Evaluate **addition** and **subtraction**, from left to right.

Example 1

$30 + 6 \div 2 - 5 \times (9 - 7)$
$= 30 + 6 \div 2 - 5 \times 2$
$= 30 + 3 - 10$
$= 23$

Example 2

$20 - (8 + (1 + 2)^2)$
$= 20 - (8 + 3^2)$
$= 20 - (8 + 9)$
$= 20 - 17$
$= 3$

Terminology

Number

Digits and Integers

A **digit** is any one of the ten numerals **0, 1, 2, 3, 4, 5, 6, 7, 8, 9**.

For example, **358** is a three-digit number.

Whole numbers: { 0, 1, 2, 3, ... }.

Counting numbers, or **Positive Integers**: { 1, 2, 3, ... }.

Integers: { ..., −2, −1, 0, 1, 2, 3, ... }.

Factors and Divisibility

Suppose $A = B \times C$, and A, B, and C are all counting numbers.
For example, 6 = 2 × 3.

- Then, A is divisible by B, and A is a multiple of B.
  6 is divisible by 2; 6 is a multiple of 2.
- Likewise, A is divisible by C, and A is a multiple of C.
  6 is divisible by 3; 6 is a multiple of 3.
- Both B and C are factors of A.
  2 and 3 are factors of 6.

A **prime number** is a counting number greater than 1, that is divisible only by 1 and itself.

Fractions

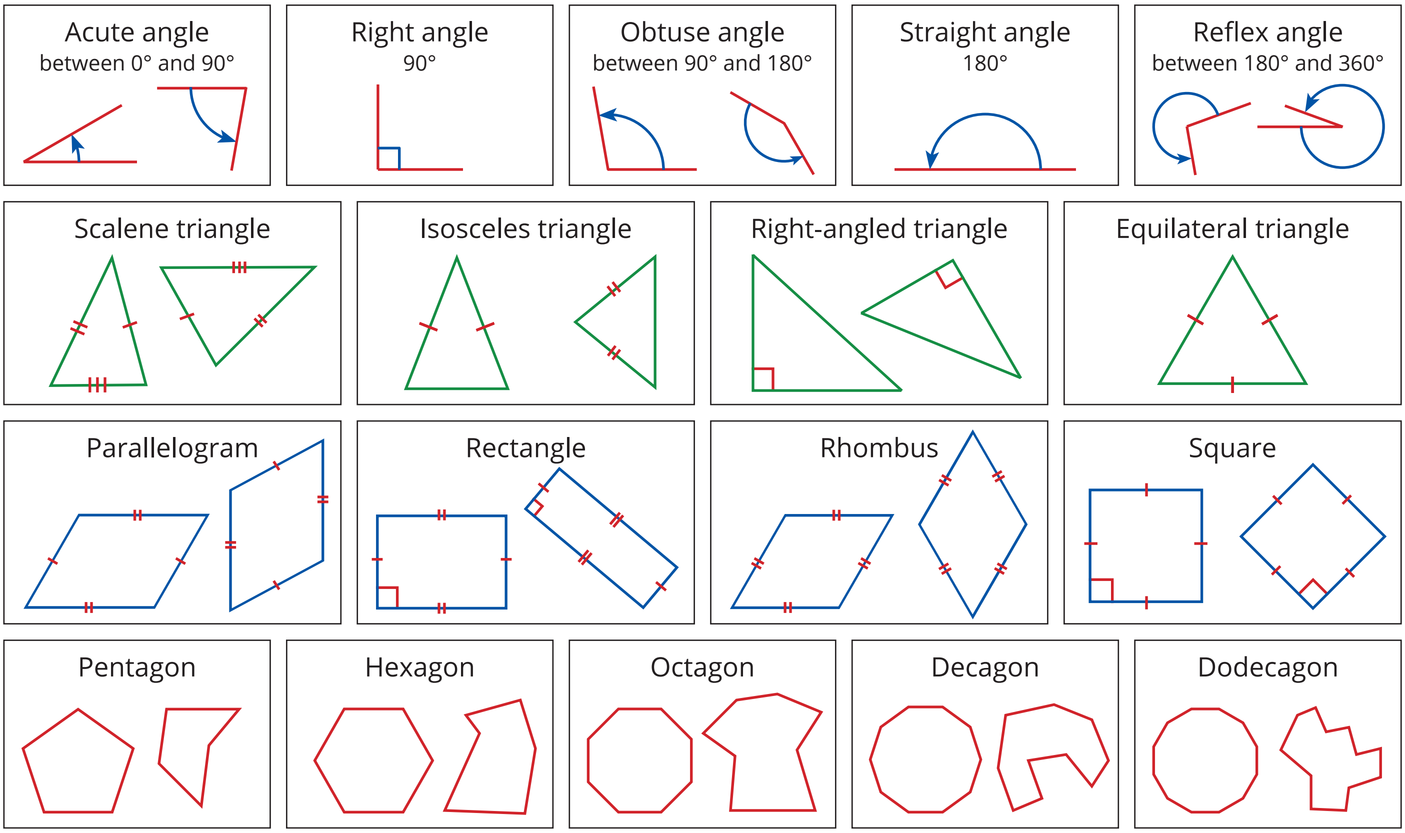
For **common** or **simple fractions** $\frac{a}{b}$, a (the **numerator**) and b (the **denominator**) are both integers, and $b \neq 0$.

In a **unit fraction**, the numerator is 1.
$\frac{1}{2}$ and $\frac{1}{100}$ are both unit fractions.

In a **proper fraction**, $a < b$.
$\frac{1}{2}$ and $\frac{5}{6}$ are both proper fractions.

In an **improper fraction**, $a \geq b$.
$\frac{3}{2}$ and $\frac{11}{8}$ are both improper fractions.

Geometry



Measures of centre

- The **mean**, or **average**, of a set of N values is the sum of the N values, divided by N.
- The **median** is the value that's exactly in the middle of the set when it is ordered.
  If there are an even number of values then the median is the mean of the two middle values.
- The **mode** is the value that occurs the greatest number of times.