APSMO

2023 : DIVISION S

WEDNESDAY 6 SEPTEMBER 2023

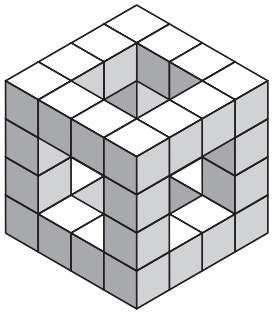
OLYMPIAD 4

Total Time Allowed: ***30 Minutes***

4A. What is the greatest possible result of dividing two different integers from the set {−7, −3, −2, +3, +4, +8} ?

Express your answer as an improper fraction or mixed numeral.

4B. A cube with 4cm edges has three channels cut through its centre, from one face to its opposite face, parallel to the faces of the original cube.



Each channel has a 2 cm × 2 cm square cross-section.

How many cubic centimetres are in the volume of the hollowed-out solid?

4C. Mr MacIntosh buys and sells apple pies.

He pays $8 for 5 apple pies.

He then sells them at 3 apple pies for $10.

Last weekend, he bought *P* apple pies and sold all of them, making a profit of $78.

Find *P*.

4D. Find the smallest integer x such that x, $x + 6$, and $2x - 9$ can represent the lengths of the sides of a triangle.

4E. In a cryptarithm, different letters represent different digits, and no leading digit equals 0.

Find the smallest possible whole number value of *GEEE*.

(Note that the letter *I* does not necessarily represent the digit 1, and the letter *O* does not necessarily represent the digit 0.)

$$\begin{array}{cccc} & F & E & E \\ + & F & I & E \\ + & F & O & E \\ + & F & U & M \\ \hline G & E & E & E \end{array}$$

Write your answers in the boxes on the back.

←

Keep your answers hidden by folding backwards on this line.

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4A.

4B.

4C.

4D.

4E.

Student Name:

Fold here. Keep your answers hidden.

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Solutions and Answers

(Items in parentheses are not required)

For teacher use only. Not for Distribution.

4A: $\frac{7}{2}$ or $3\frac{1}{2}$	**4B:** 32 (cm^3)	**4C:** 45	**4D:** 8	**4E:** 1444

4A. The question is: Find the greatest possible result of dividing two different integers from the set {−7, −3, −2, +3, +4, +8}.

Multiplying and Dividing with Negative Numbers

Using equivalent number sentences to demonstrate the relationship between multiplication and division:

$2 \times 3 = 3 + 3$
$= 6$
$6 \div 2 = 3$ -(1)

We can reason that:

$2 \times -3 = -3 + -3$
$= -6$
$-6 \div 2 = -3$ -(2)

and

$-2 \times 3 = -2 + -2 + -2$
$= -6$
$-6 \div -2 = 3$ -(3)

We can also reason that:

$-2 \times 1 = -2$
$-2 \times (4 - 3) = -2$
$(-2 \times 4) + (-2 \times -3) = -2$
$-8 + (-2 \times -3) = -2$
$-2 \times -3 = -2 + 8$
$-2 \times -3 = 6$
$6 \div -2 = -3$ -(4)

METHOD: *Eliminate all but one possibility.*

From the above reasoning, we know that:

(1) Positive number ÷ Positive number = Positive number

(2) Negative number ÷ Positive number = Negative number

(3) Negative number ÷ Negative number = Positive number

(4) Positive number ÷ Negative number = Negative number

Given that there are positive results, there is no need to calculate any results that are negative.

We can also ignore dividends with small magnitudes, and divisors with large magnitudes, since we want our fractional result to have a large numerator and a small denominator.

Divisor \ Dividend	−7	−3	−2	+3	+4	+8
−7		$\frac{3}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$	$-\frac{4}{7}$	$-\frac{8}{7}$
−3	$\frac{7}{3}$		$\frac{2}{3}$	−1	$-\frac{4}{3}$	$-\frac{8}{3}$
−2	$\frac{7}{2}$	$\frac{3}{2}$		$-\frac{3}{2}$	−2	−4
+3	$-\frac{7}{3}$	-1	$-\frac{2}{3}$		$\frac{4}{3}$	$\frac{8}{3}$
+4	$-\frac{7}{4}$	$-\frac{3}{4}$	$-\frac{1}{2}$	$\frac{3}{4}$		2
+8	$-\frac{7}{8}$	$-\frac{3}{8}$	$-\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	

The two possibilities are $\frac{7}{2}$ and $\frac{8}{3}$. We can express them both as mixed numerals: $\frac{7}{2} = 3\frac{1}{2}$ and $\frac{8}{3} = 2\frac{2}{3}$.

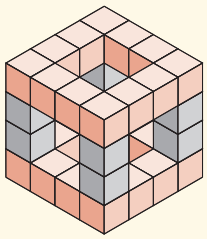
The greatest result is $\mathbf{\frac{7}{2}}$ **or** $\mathbf{3\frac{1}{2}}$.

***Follow-Up**: Find the greatest possible product when multiplying 3 different numbers from the given set.* [168]

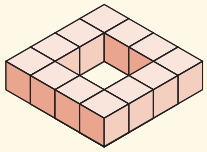
4B. The question is: How many cubic centimetres are in the volume of the hollowed-out solid?

METHOD 1: *Consider each 1-cm-thick layer separately.*

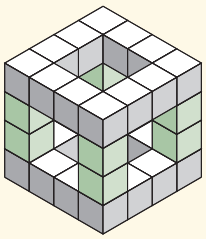
The top layer and the bottom layer both have the same arrangement of cubes.



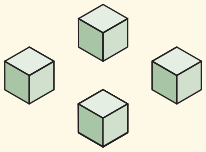
Both layers comprise 12 cubes each.



The two middle layers both have the same arrangement of cubes.



Both layers comprise 4 cubes each.



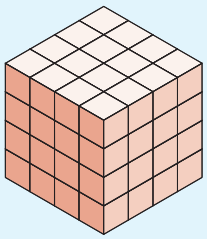
There are two layers with 12 cubes each, and two layers with 4 cubes each.

Each cube has a volume of $1\,\text{cm}^3$.

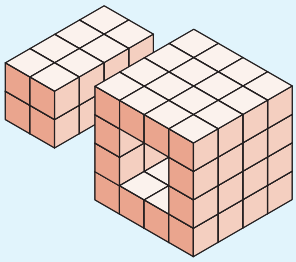
In total, the solid has a volume of $12 + 4 + 4 + 12 = \mathbf{32}\,\text{cm}^3$.

METHOD 2: *Subtract the cubes that were removed (1).*

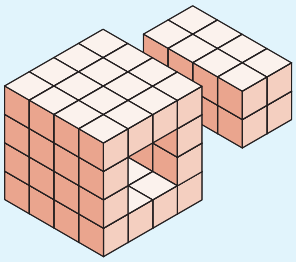
The original cube with $4\,\text{cm}$ edges has a volume of $4 \times 4 \times 4 = 64\,\text{cm}^3$.



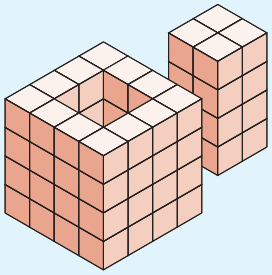
The channel from front to back has a volume of $2 \times 2 \times 4 = 16\,\text{cm}^3$.



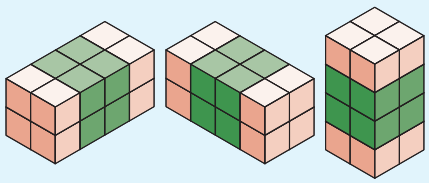
Likewise, the channel from left to right has a volume of $16\,\text{cm}^3$,



as does the channel from top to bottom.



All three channels share the same central $2 \times 2 \times 2 = 8\,\text{cm}^3$ cube.

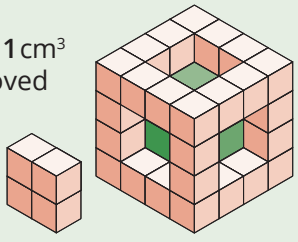


When forming the channels, the central cube is "cut out" three times. We can compensate for this by adding it back in twice.

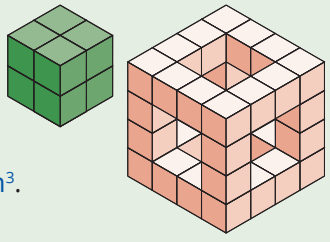
The remaining solid has a volume of $64 - 16 - 16 - 16 + 8 + 8 = \mathbf{32}\,\text{cm}^3$.

METHOD 3: *Subtract the cubes that were removed (2).*

Beginning with a $64\,\text{cm}^3$ cube, four $1\,\text{cm}^3$ cubes were removed from each of the six faces,



and a single $2 \times 2 \times 2 = 8\,\text{cm}^3$ cube was removed from the centre.



The remaining solid has a volume of $64 - (6 \times 4) - 8 = \mathbf{32}\,\text{cm}^3$.

***Follow-Up**: How many square centimetres are in the surface area of the final shape?* [120]

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4C. The question is: Find P, the number of apple pies that Mr MacIntosh bought to make a profit of \$78.

METHOD 1 ***Strategy:*** *Build a table.*

Mr MacIntosh buys multiples of 5 apple pies: 5 apple pies for \$8.

Batches of 5 pies bought	1	2	3	4	5	6	...
Total number of pies	5	10	15	20	25	30	...
Mr MacIntosh's cost price	\$8	\$16	\$24	\$32	\$40	\$48	...

Since he sells all of the pies at 3 apple pies for \$10, the number of pies must also be a multiple of 3.

Batches of 5 pies bought	1	2	3	4	5	6	...
Total number of pies	5	10	15	20	25	30	...
Mr MacIntosh's cost price	\$8	\$16	\$24	\$32	\$40	\$48	...
Batches of 3 pies sold			5			10	...
Mr MacIntosh's sale price			\$50			\$100	...

His profit can be found by subtracting his cost price from his selling price.

To make \$78 profit, Mr MacIntosh would have bought **45** pies.

Batches of 5 pies bought			3			6			9
Total number of pies			15			30			45
Mr MacIntosh's cost price			\$24			\$48			\$72
Batches of 3 pies sold			5			10			15
Mr MacIntosh's sale price			\$50			\$100			\$150
Profit	\$50 – \$24 = \$26			\$100 – \$48 = \$52			\$150 – \$72 = \$78		

METHOD 2 ***Strategy:*** *Find the profit for 3 pies.*

Mr MacIntosh bought 5 apple pies for \$8.

Each pie costs him \$8 ÷ 5 = \$1.60.

3 pies would cost \$1.60 × 3 = \$4.80.

Since Mr MacIntosh sells 3 apple pies for \$10, his profit for every 3 pies is \$10 – \$4.80 = \$5.20.

Mr MacIntosh's total profit is \$78.

Pies	3	6	9	12	15	30	45
Profit	\$5.20	\$10.40	\$15.60	\$20.80	\$26	\$52	\$78

a whole number of dollars

Mr MacIntosh bought **45** pies to make \$78 profit.

METHOD 3 ***Strategy:*** *Find the profit for 1 pie.*

The profit for 1 pie is $\frac{10}{3} - \frac{8}{5} = \frac{50}{15} - \frac{24}{15} = \frac{26}{15}$ dollars.

The profit for P pies is 78 dollars.

We can express Mr MacIntosh's weekend earnings using the equation $\frac{26}{15}P = 78$.

	$\frac{26}{15}P = 78$
Multiplying both sides by 15:	$26P = 1170$
Dividing both sides by 26:	$P = 45$

Mr MacIntosh bought **45** pies.

Follow-Up: *The next weekend, Mr MacIntosh made a profit of \$170. What is the smallest number of pies he could have sold?* [99. He had to purchase 100 pies for \$160, to be able to sell 99 pies for \$330.]

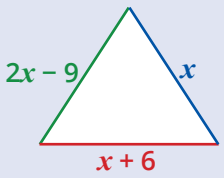
4D. The question is: Find the smallest integer x such that x, $x + 6$, and $2x - 9$ can represent the lengths of the sides of a triangle.

METHOD 1 ***Strategy:*** *Build a table, and draw a diagram.*

We want the smallest possible integer value for x.

Since it is a side length, x must be no less than 1.

x	1	2	3	4	5	6	7	8
$x + 6$	7	8	9	10	11	12	13	14
$2x - 9$	-7	-5	-3	-1	1	3	5	7



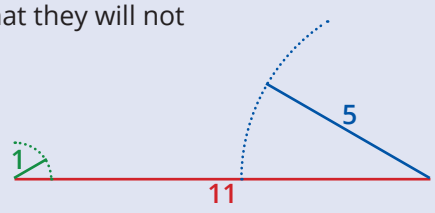
We can see that values of x that are less than 5 are impossible, since they will result in a negative side length for one side of the triangle.

Suppose $x = 5$. The other two side lengths would then be 11 and 1.

To construct this triangle, we might begin with a base that is 11 units long.

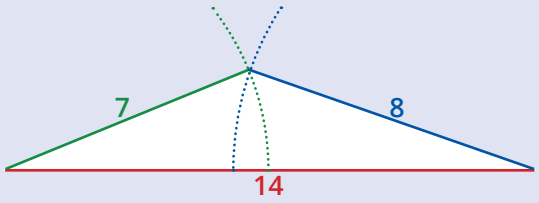
However, when we fill in the two sides with lengths 5 and 1, we can see that they will not meet in the middle.

It is not possible to have a triangle with side lengths 5, 11, and 1.



For the two shorter sides to meet in the middle, their combined length must be greater than the length of the longest side.

From our table, we can see that the smallest value of x for which this occurs is $x = 8$. The other two side lengths would be 14 and 7.



The smallest integer x would be **8**.

METHOD 2 ***Strategy:*** *Reason algebraically.*

Recognising that all side lengths must be positive, and the combined length of any two sides must be greater than the length of the third side, we can construct the following inequalities:

$x + x + 6 > 2x - 9$	$x + 2x - 9 > x + 6$	$x + 6 + 2x - 9 > x$
$2x + 6 > 2x - 9$	$2x - 9 > 6$	$2x - 3 > 0$
$6 > -9$	$2x > 15$	$2x > 3$
This is a true statement, but not useful for solving our problem.	$x > 7.5$ -(1)	$x > 1.5$ -(2)

The two conditions that must be satisfied are that $x > 7.5$, and $x > 1.5$.

The smallest integer value of x for which both conditions work is $x = 8$.

Follow-Up: *Find the smallest whole number x such that x, $x + 2$, $2x$, and 23 could be the lengths of the sides of a quadrilateral.* [6]

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4E. The question is: Find the smallest possible whole number value of *GEEE*.

	F	*E*	*E*	
+	*F*	*I*	*E*	
+	*F*	*O*	*E*	
+	*F*	*U*	*M*	
G	*E*	*E*	*E*	

METHOD: *Eliminate all but one possibility.*

We know that:

- The leading digit for any number cannot be 0, so neither ***F*** nor ***G*** can be 0.
- In the ones column, ***E*** + ***E*** + ***E*** + ***M*** gives a result that has ***E*** in the ones place. It is therefore impossible for the value of ***E*** to be 0.

We want the least possible whole number value of ***GEEE***, so let's begin by trying ***G*** = 1.

If ***G*** = 1, then ***E*** cannot be 1, so we'll try ***E*** = 2.

	F	2	2	
+	*F*	*I*	2	
+	*F*	*O*	2	
+	*F*	*U*	*M*	
1	2	2	2	

If ***G*** = 1 and ***E*** = 2, then ***F*** cannot be 0, 1 or 2.

The only possible value for ***F*** is 3, since values greater than 3 will result in a larger value for ***GEEE***.

	3	2	2	
+	3	*I*	2	
+	3	*O*	2	
+	3	*U*	*M*	
1	2	2	2	

2 + 2 + 2 + ***M*** = 6 + ***M*** gives a result that has 2 in the ones place.

M must then be equal to 6, with regrouping to the tens place.

		1		
	3	2	2	
+	3	*I*	2	
+	3	*O*	2	
+	3	*U*	6	
1	2	2	2	

In the tens column,
1 + 2 + ***I*** + ***O*** + ***U*** = 2, since we cannot have any regrouping to the hundreds place.

Such a result is not possible.

		1		
	3	2	2	
+	3	*I*	2	
+	3	*O*	2	
+	3	*U*	6	
1	2	2	2	

Let's try the next smallest possible whole number value of ***GEEE***.

We cannot have ***G*** = 1 and ***E*** = 3, since this will only work if ***F*** = 3, and we cannot have two letters that both represent 3.

	F	3	3	
+	*F*	*I*	3	
+	*F*	*O*	3	
+	*F*	*U*	*M*	
1	3	3	3	

The next smallest possible value for ***GEEE*** is 1444.

In this case, it is possible for the value of ***F*** to be 3.

There will need to be regrouping of two hundreds from the tens place.

	2			
	3	4	4	
+	3	*I*	4	
+	3	*O*	4	
+	3	*U*	*M*	
1	4	4	4	

4 + 4 + 4 + ***M*** = 12 + ***M*** gives a result that has 4 in the ones place.

M must then be equal to 2, with regrouping to the tens place.

	2	1		
	3	4	4	
+	3	*I*	4	
+	3	*O*	4	
+	3	*U*	2	
1	4	4	4	

In the tens column,
1 + 4 + ***I*** + ***O*** + ***U*** = 24, since we need to regroup two hundreds to the hundreds place.

This can occur if, for example, ***I*** = 5, ***O*** = 6 and ***U*** = 8.

	2	1		
	3	4	4	
+	3	5	4	
+	3	6	4	
+	3	8	2	
1	4	4	4	

The smallest possible whole number value of *GEEE* is **1444.**

Follow-Up: *In the given cryptarithm, find the greatest possible value of GEEE.* [3888; one possible solution is 988 + 978 + 968 + 954 = 3888]