APSMO

2021 : DIVISION S

WEDNESDAY 24 MARCH 2021

OLYMPIAD 1

Total Time Allowed: ***30 Minutes***

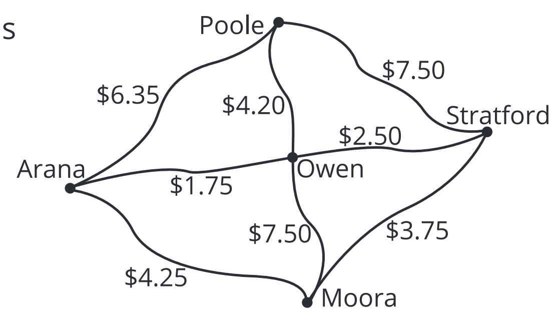
Write your answers in the boxes on the back.

Keep your answers hidden by folding backwards on this line.

1A. Calculate: $\dfrac{13^2 - 12^2}{13 + 12}$

1B. The roadmap shows the tolls when travelling between Arana, Poole, Stratford, Moora, and Owen.

What is the sum of the tolls along the least expensive route to travel from Poole to Moora?



1C. The six faces of a number cube are labelled 1, 2, 3, 4, 5, and 6.

In how many different ways can a pair of numbered faces be selected so that they share a common edge?

[Note: The face pair 3-1 is the same as 1-3, and counts as one pairing.]

1D. A piece of string is cut to be 75% of its original length.

Then, the string is again cut to be 75% of the new length, and so on, with each cut making the string 75% of its previous length.

After 22 cuts, the string is 3 cm long.

How long was the string, in centimetres, after 20 cuts?

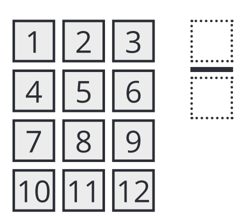
Express your answer as a fraction in lowest terms.

1E. Fractions are formed using a set of cards labelled 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, by placing one card above the other, as shown in the diagram.

How many different fractions greater than $\frac{1}{3}$ but less than 1 can be formed using these cards?

Consider equivalent fractions as different fractions.

[E.g., $\frac{2}{4}$ and $\frac{6}{12}$ have the same value, but are considered different in this question.]



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OLYMPIAD 1

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Student Name:

1A.

1B.

1C.

1D.

1E.

Fold here. Keep your answers hidden.

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MATHS OLYMPIAD

OLYMPIAD 1

Solutions and Answers

For teacher use only. Not for Distribution.

1A: 1	**1B:** \$10.20	**1C:** 12	**1D:** $\frac{16}{3}$ or $5\frac{1}{3}$	**1E:** 44

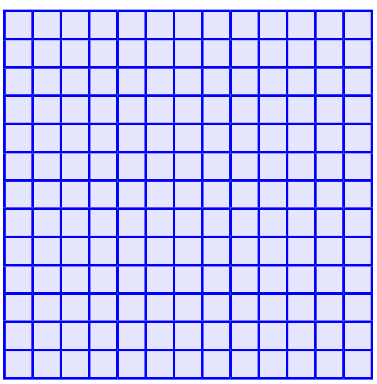
1A. The question is: Calculate $\frac{13^2 - 12^2}{13 + 12}$.

METHOD 1 ***Strategy:*** *Perform the calculation.*

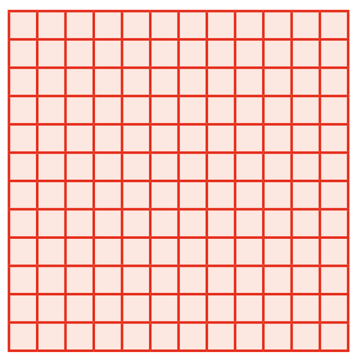
Since **13 × 13 = 169**, and **12 × 12 = 144**, we have: $\frac{13^2 - 12^2}{13 + 12} = \frac{169 - 144}{13 + 12} = \frac{25}{25} = \mathbf{1}$.

METHOD 2 ***Strategy:*** *Draw a diagram.*

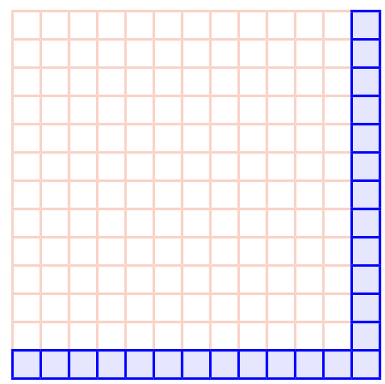
We can represent 13^2 as an array with **13** columns and **13** rows.



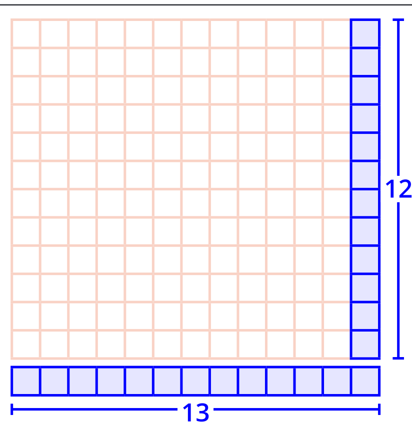
Likewise, 12^2 can be represented as an array with **12** columns and **12** rows.



So $13^2 - 12^2$ is equivalent to removing the **12 × 12** array from the **13 × 13** array.



After removing the **12 × 12** array, what remains can be thought of as a column of **12** and a row of **13**, for a total of **12 + 13 = 25**.



Therefore,

$$13^2 - 12^2 = 12 + 13 = 25$$

and so

$$\frac{13^2 - 12^2}{13 + 12} = \frac{25}{25} = \mathbf{1}.$$

* Note that a more sophisticated version of this solution would see the students recognising that $13^2 - 12^2$ is the difference of two squares, and so can be factorised $13^2 - 12^2 = (13 + 12)(13 - 12)$.

Follow-Up: *Evaluate* $\frac{4^4 - 3^4}{3^2 + 4^2}$. [7]

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OLYMPIAD 1

1B. The question is: What is the sum of the tolls along the least expensive route to travel from Poole to Moora?

METHOD 1 ***Strategy:*** *Make an organised list of all possible paths from Poole to Moora.*

Listing the paths alphabetically, we have:

P – A – M	6.35 + 4.25 = 10.60
P – A – O – M	6.35 + 1.75 + 7.50 = 15.60
P – A – O – S – M	6.35 + 1.75 + 2.50 + 3.75 = 14.35
P – O – A – M	4.20 + 1.75 + 4.25 = 10.20
P – O – M	4.20 + 7.50 = 11.70
P – O – S – M	4.20 + 2.50 + 3.75 = 10.45
P – S – M	7.50 + 3.75 = 11.25
P – S – O – A – M	7.50 + 2.50 + 1.75 + 4.25 = 16.00
P – S – O – M	7.50 + 2.50 + 7.50 = 17.50

The least expensive route from Poole to Moora costs **$10.20**.

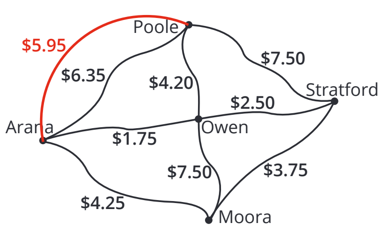
METHOD 2 ***Strategy:*** *Consolidate paths to find the least expensive route between towns.*

Poole to Arana direct costs **$6.35**.

Going via Owen, it costs **$4.20 + $1.75 = $5.95**, which is cheaper than **$6.35**.

Let's indicate that there is a less expensive route.

It costs $5.95 to travel from Poole to Arana.

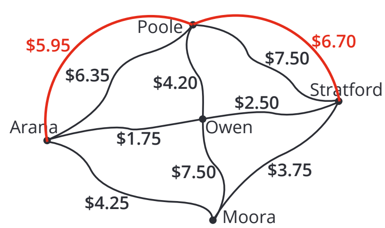


Poole to Stratford direct costs **$7.50**.

Going via Owen, it costs **$4.20 + $2.50 = $6.70**, which is cheaper than **$7.50**.

Let's indicate that there is a less expensive route.

It costs $6.70 to travel from Poole to Stratford.

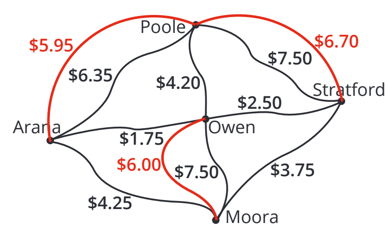


Owen to Moora direct costs **$7.50**.

Going via Arana, it costs **$1.75 + $4.25 = $6.00**.

Going via Stratford, it costs **$2.50 + $3.75 = $6.25**.

The least expensive route from Owen to Moora costs $6.00.



So, for Poole to Moora:

Going via Arana, it costs $5.95 + **$4.25 = $10.20**.

Going via Owen, it costs **$4.20** + $6.00 = **$10.20**.

Going via Stratford, it costs $6.70 + **$3.75 = $10.45**.

Therefore the least expensive route from Poole to Moora costs **$10.20**.

Follow-Up: *Find the cost of the least expensive path that starts at Arana and ends at Owen, and includes travelling to all five locations exactly once.* [$19.70: Arana - Moora - Stratford - Poole - Owen]

1C. The question is: In how many different ways can a pair of numbered faces be selected so that they share a common edge?

METHOD 1 ***Strategy:*** *Subtract the number of opposite pairs from the total number of pairs.*

Each face on the cube can be paired with one of **5** other faces.

So there are **6 × 5 = 30** possible ways to list two faces.

	1	2	3	4	5	6
1		1-2	1-3	1-4	1-5	1-6
2	2-1		2-3	2-4	2-5	2-6
3	3-1	3-2		3-4	3-5	3-6
4	4-1	4-2	4-3		4-5	4-6
5	5-1	5-2	5-3	5-4		5-6
6	6-1	6-2	6-3	6-4	6-5	

Since **1-3** and **3-1** are considered the same, this double-counts each pairing.

Thus there are only **30 ÷ 2 = 15** pairings.

	1	2	3	4	5	6
1		1-2	1-3	1-4	1-5	1-6
2	~~2-1~~		2-3	2-4	2-5	2-6
3	~~3-1~~	~~3-2~~		3-4	3-5	3-6
4	~~4-1~~	~~4-2~~	~~4-3~~		4-5	4-6
5	~~5-1~~	~~5-2~~	~~5-3~~	~~5-4~~		5-6
6	~~6-1~~	~~6-2~~	~~6-3~~	~~6-4~~	~~6-5~~	

For each face, there is one face that is on the other side of the cube, and so they do not share an edge.

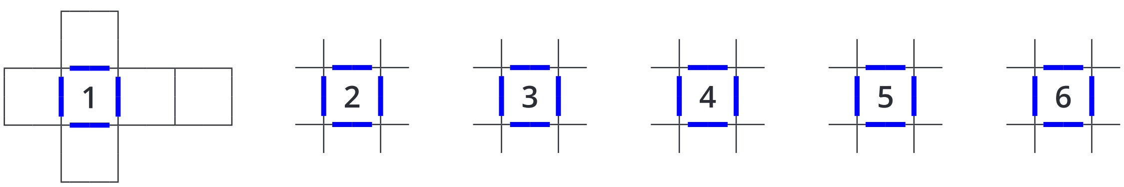
There are **3** such pairings.

It does not matter how the faces are arranged; but to simplify the explanation, we can imagine that the cube is configured like a standard die, with opposite faces adding up to 7.

	1	2	3	4	5	6
1		1-2	1-3	1-4	1-5	1-6
2	~~2-1~~		2-3	2-4	2-5	2-6
3	~~3-1~~	~~3-2~~		3-4	3-5	3-6
4	~~4-1~~	~~4-2~~	~~4-3~~		4-5	4-6
5	~~5-1~~	~~5-2~~	~~5-3~~	~~5-4~~		5-6
6	~~6-1~~	~~6-2~~	~~6-3~~	~~6-4~~	~~6-5~~	

Therefore there are **15 – 3 = 12** ways to select a pair of faces that share a common edge.

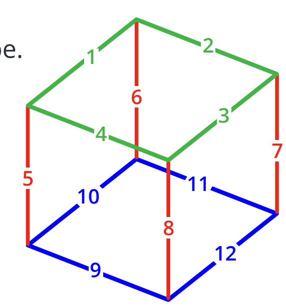
METHOD 2 ***Strategy:*** *Use the idea that each face is adjacent to 4 other faces.*



Since order does not matter, the number of pairings is $\frac{6 \times 4}{2}$ = **12**.

METHOD 3 ***Strategy:*** *Consider the number of edges that form a cube.*

There are 12 edges on a cube.



Each edge joins 2 adjacent faces.

Therefore, there must be **12** pairings.

Follow-Up: *Isabel rolls two number cubes and finds the product of their faces. How many different products are possible?* [18]

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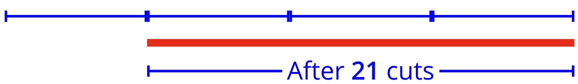
OLYMPIAD 1

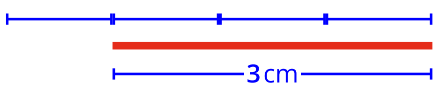
1D. The question is: How long was the string, in centimetres, after 20 cuts?

METHOD 1 ***Strategy:*** *Work backwards.*

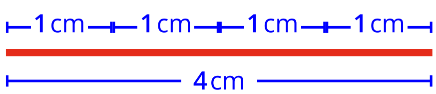
Suppose the string is this long after **20** cuts.



After **21** cuts, the string is $\frac{3}{4}$ of the length after **20** cuts.



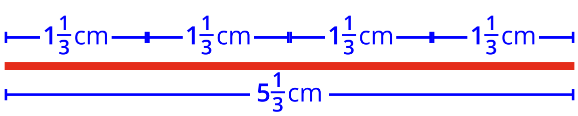
After **22** cuts, the string is $\frac{3}{4}$ of the length after **21** cuts.
The string is now **3** cm long.



For the **22**nd cut, the piece that was cut off was $\frac{1}{3}$ of the resulting length of **3** cm.

$\frac{1}{3} \times 3\text{ cm} = 1\text{ cm}$.

So the length after **21** cuts was **3** cm + **1** cm = **4** cm.



For the **21**st cut, the piece that was cut off was $\frac{1}{3}$ of the resulting length of **4** cm.

$\frac{1}{3} \times 4\text{ cm} = 1\frac{1}{3}\text{ cm}$.

So the length after **20** cuts was **4** cm + $1\frac{1}{3}$ cm = $\mathbf{5\frac{1}{3}}$ cm.

METHOD 2 ***Strategy:*** *Use algebra.*

Let x be the length after **20** cuts.

After **21** cuts, the length is $\frac{3}{4}$ of x, or $\frac{3x}{4}$.

After **22** cuts, the length is $\frac{3}{4}$ of $\frac{3x}{4}$, or $\frac{9x}{16}$.

Therefore, $\frac{9x}{16} = 3\text{ cm}$.

$$\frac{9x}{16} = 3\text{ cm}$$

$$\frac{9x}{16} \times 16 = 3\text{ cm} \times 16$$

$$9x = 48\text{ cm}$$

$$x = \frac{48}{9}\text{ cm}$$

Therefore $x = \frac{48}{9}$ cm

$= \frac{16}{3}$ cm.

The string must have been $\frac{16}{3}$ cm long after **20** cuts.

Follow-Ups: *How many cuts were performed before the string was less than 10 cm long?* [18]

How many cuts will be performed before the string is less than 1 cm long? [26]

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OLYMPIAD

1

1E. The question is: How many different fractions greater than $\frac{1}{3}$ but less than 1 can be formed?

Strategy: *Make an organised list.*

We can create an organised list based on either the numerator or the denominator.

METHOD 1: *List by denominator.*

Since the fractions must be less than 1, we can list all of the fractions for each denominator in order, until the numerator is equal to the denominator.

Increasing the denominator reduces the value of the fraction, so we can:

- Locate fractions that are equivalent to $\frac{1}{3}$, then
- Only count fractions that have a greater value.

	Count
$\frac{1}{2}, \frac{2}{2}$	1
$\frac{1}{3}, \frac{2}{3}, \frac{3}{3}$	1
$\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}$	2
$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}$	3
$\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6}$	3
$\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{7}{7}$	4
$\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, \frac{8}{8}$	5
$\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9}, \frac{9}{9}$	5
$\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}, \frac{10}{10}$	6
$\frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \frac{4}{11}, \frac{5}{11}, \frac{6}{11}, \frac{7}{11}, \frac{8}{11}, \frac{9}{11}, \frac{10}{11}, \frac{11}{11}$	7
$\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}, \frac{7}{12}, \frac{8}{12}, \frac{9}{12}, \frac{10}{12}, \frac{11}{12}, \frac{12}{12}$	7
Total	44

44 fractions between $\frac{1}{3}$ and 1 can be formed using these cards.

METHOD 2: *List by numerator.*

We can begin with fractions that are equivalent to $\frac{1}{3}$.

From these, we can construct fractions of increasing value by decreasing the denominator.

	Count
$\frac{1}{3}, \frac{1}{2}, \frac{1}{1}$	1
$\frac{2}{6}, \frac{2}{5}, \frac{2}{4}, \frac{2}{3}, \frac{2}{2}$	3
$\frac{3}{9}, \frac{3}{8}, \frac{3}{7}, \frac{3}{6}, \frac{3}{5}, \frac{3}{4}, \frac{3}{3}$	5
$\frac{4}{12}, \frac{4}{11}, \frac{4}{10}, \frac{4}{9}, \frac{4}{8}, \frac{4}{7}, \frac{4}{6}, \frac{4}{5}, \frac{4}{4}$	7
$\frac{5}{12}, \frac{5}{11}, \frac{5}{10}, \frac{5}{9}, \frac{5}{8}, \frac{5}{7}, \frac{5}{6}, \frac{5}{5}$	7
$\frac{6}{12}, \frac{6}{11}, \frac{6}{10}, \frac{6}{9}, \frac{6}{8}, \frac{6}{7}, \frac{6}{6}$	6
$\frac{7}{12}, \frac{7}{11}, \frac{7}{10}, \frac{7}{9}, \frac{7}{8}, \frac{7}{7}$	5
$\frac{8}{12}, \frac{8}{11}, \frac{8}{10}, \frac{8}{9}, \frac{8}{8}$	4
$\frac{9}{12}, \frac{9}{11}, \frac{9}{10}, \frac{9}{6}$	3
$\frac{10}{12}, \frac{10}{11}, \frac{10}{10}$	2
$\frac{11}{12}, \frac{11}{11}$	1
$\frac{12}{12}$	0
Total	44

It is possible to construct **44** different fractions with values between $\frac{1}{3}$ and 1.

Follow-Up: *If equivalent fractions are considered the same, how many unique values would there be between $\frac{1}{3}$ and 1 non-inclusive?* [30]