Maths Olympiad Contest Problems

Volume 3

Australian Edition

Exploring Maths Through Problem Solving

Contains APSMO Maths Olympiad Papers From 2006 to 2013

Contents

Preface to Australian Edition

Australasian Problem Solving Mathematical Olympiads (APSMO) Inc is proud to be affiliated with Mathematical Olympiads for Elementary and Middle Schools (MOEMS).

APSMO has been providing Mathematical Olympiads to schools throughout Australia and New Zealand since 1987. Our annual interschool Olympiads are held five times a year between May and September. There are two Divisions in the Olympiads, Division J for students up to 12 years of age and in school Year 6 or below, and Division S for students up to 14 years of age and in school Year 8 or below.

This book is the third volume to Maths Olympiad Contest Problems for Primary and Middle Schools (Australian Edition), containing the past Olympiad questions from APSMO Olympiads held between 2006 and 2013. It is an excellent resource, good for review and practice of problem solving and working mathematically techniques.

We take this opportunity to thank MOEMS for permission to reprint this text with the following modifications:

- Australian spelling
- Changes in nomenclature such as imperial to decimal measurements, American coinage to Australian coinage
- All Olympiad questions remain true to the original. In certain situations the answers may differ to the original answers, however all care has been taken to ensure that the purpose and solution methods remain unchanged. Consequently, we have continued to use 1c and 2c coins although they are no longer in use in Australia.
- Where it was not possible to change a question without altering the solution methods or intention of the question, a note has been included within the question text as clarification for students. For example: [Note: There are 3 feet in 1 yard].

Thank you to Dr Heather McMaster, lecturer in mathematics education at the University of Sydney, for her valuable assistance in reviewing the alterations and ensuring that the modified questions contained within this text are correct and suitable for Australian students.

Jonathan Phegan
March, 2015

Contest Problem Types

Many but not all contest problems can be categorised. This is useful if you choose to work with several related problems even if they involve different concepts.

KEY: problems are organised by type and are coded by **page number and problem placement** on that page. For example, "**Long division: 36BD, 84A, 165E**" refers to four questions, each involving long division: problems B and D on page 36, problem A on page 84, and problem E on page 165.

Note: Pages 24-64 refer to Division J contest problems and pages 66-106 refer to Division S contest problems. Each *FOLLOW-UP* problem is located after a model solution to a contest problem, is related conceptually to it, and usually extends or expands an aspect of it.

Introduction

For the Reader

This book was written for both the participants in the Australasian Problem Solving Mathematical Olympiads and their advisors. It is suitable for mathletes who wish to prepare well for the contests, students who wish to develop higher-order thinking, and teachers who wish to develop more capable students. All problems were designed to help students develop the ability to think mathematically, rather than to teach more advanced or unusual topics. While a few problems can be solved using algebra, nearly all problems can be solved by other, more elementary, methods. In other words, the fun is in devising non-technical ways to solve each problem.

The 400 Math Olympiad contest problems contained in this book are organised into 16 sets of five contests each. Every set represents one year's competition. The first eight sets were created for Division J, for students in years 4-6, and the other eight for Division S, for students in years 7-8. These problems exhibit varying degrees of difficulty and were written for contests from 2006 to 2013, inclusive.

The introduction is arranged into three parts. Sections 1 through to 5, written for all readers, contain discussions of problem solving in general. Sections 6 through to 8 offer many suggestions for getting the most out of this book. Sections 9 through to 14, designed for the advisor, called the Person-In-Charge-of-the-Olympiads (PICO), include recommendations related to the various aspects of organising a Maths Olympiad program.

2. Why We Study Problem Solving

Most people, including children, love puzzles and games. It is fun to test ourselves against challenges. The continuing popularity of crossword, jigsaw, and sudoku puzzles, as well as of board, card, and video games, attests to this facet of human nature.

Problem solving builds on this foundation. A good problem is engaging in both senses of the word. Child or adult, we readily accept the challenge, wanting to prove to ourselves that "I can do this." To many of us, a problem is fun more than it is work. A good problem captures our interest, and once you have our interest, you have our intelligence.

A good problem contains within it the promise of the thrill of discovery, that magic "Aha!" moment. It promises that deep satisfaction, if we solve it, of knowing we've accomplished something. It promises growth, the realisation that we will know more today than we did yesterday. It speaks to our universal desire for mastery. Babies reach continually for their toes until they succeed in grasping them. Toddlers fall continually until they succeed in walking. Children swing continually at a baseball until they succeed in hitting it frequently. A good problem promises us many things, all of them worthwhile.

The history of mathematics is a history of problem-solving efforts.

It is said that perhaps all mathematics evolved as the result of problem solving. Often, one person challenged another, or perhaps himself, to handle an unexpected question. Sometimes the solution extended the range of knowledge in a well-known field or even led to the creation of a new field.

Very little of the knowledge we have today is likely to have developed in the way we study it. What we see in many books is the distilled material, whose elegant, economical organisation and presentation obscure the struggling and false starts that went into polishing it. That which grabbed and held the imagination has been scrubbed out.

A typical example of this is high school geometry. The ancient Greek mathematicians enjoyed challenging each other. "Alpha" would ask "Beta" to find a way to perform a specific construction using only a straightedge and a pair of collapsible compasses. But it wasn't enough for Beta to merely create a method. He also had to convince Alpha that his method would always work. Thus, proofs were born, as a result of problem solving. Slowly, the Greeks built up a whole body of knowledge. Euclid's genius was in collecting all the properties into an organised reference book in which most of them grew logically from a very few basic properties.

For centuries we have taught geometry according to Euclid, instead of tracing the journey taken to arrive at the knowledge. This removed the thought-provoking elements of problem solving mentioned above and with it, much of the involvement for many students.

Therefore…, why should we learn maths through problem solving? To put the magic back into maths.

For the PICO *

9. Why We *Teach* Problem Solving

"When am I ever going to use this?" Contrary to common belief, these words are rarely uttered by a student intent on discussing the future. Rather, they seem to be a statement of frustration, an indirect way of saying, "I don't know what I'm doing, and I don't like feeling uncertain." Cataloguing careers that use mathematics or listing high school maths courses does not seem likely to reassure such a student. Reassurance comes best from giving the student ownership of the skill or concept, and by helping the student become comfortable with the work.

Teachers can provide ownership by posing a good problem and then by allowing the student to find his or her own way to solve it. Teachers can use that innate love of a challenge to engage the student's interest, and therefore the student's intelligence, as discussed in Section 2. The student, responding to the challenge, sees reasons to focus on the problem. If the problem is well chosen and constructed, the student learns something important.

How does teaching through problem solving help the teacher to develop stronger students?

1. **Interest:** Good problems tap into something deep within all of us. Assigning them to students engages their interest and focus.
2. **Meaning:** Embedding a concept or skill into a good problem provides more meaning to students than merely stating or demonstrating it; students can see a purpose. Skills are not learned just for their own sake, but are seen as tools to be used, not as ends in themselves.
3. **Complexity:** Life is not simple. Life's problems are not simple. We do students no favour by making mathematical problems simple. Most good problems require more than one concept to solve. Assigning good problems may help students develop the ability to see the interplay between different concepts.
4. **Creativity and flexibility:** Many of the problems in this book show multiple solutions. The more ways a student can see to solve a problem, the freer he or she may feel to try an unusual approach to a puzzling problem. Some of the problems in this book have triggered highly inventive methods.
5. **Developing mathematical thinking:** Over time, continued exposure to thoughtful solutions leads students to think mathematically. Subsequent maths courses in high school will require this ability.
6. **Retention:** A good problem often allows for many strategies and may require several principles and skills with each strategy. Continually tackling problems and discussing solutions allows the student to revisit most concepts and procedures frequently, each time from a fresh perspective. This disguised practice builds in reinforcement while it clarifies the concept.

*** PICO: P<u>erson</u> In Charge of Olympiads**

7. **Building student confidence:** Each demanding problem the teacher assigns reflects a respect for the students' abilities, which students may well appreciate. With each successful solution, the student realises, "I can do this!" Since nothing builds self-confidence like accomplishment, these two things can stimulate intellectual growth.
8. **Empowerment:** When the teacher allows students to tackle problems on their own with no more than an occasional hint and then asks several students to present different approaches aloud, the students assume ownership of the problems. Students are likely to accept that the responsibility for learning is theirs, not the teacher's. Each student becomes an active partner in the learning process. The problem, the solution, the principles involved all become his or her property and are more likely to be available when needed.

10. Why Should We Do APSMO Contests?

As discussed in Section 2, various puzzles, games, and sports leagues are immensely popular among both children and their parents, and have been so for many generations. In fact, children seem to turn almost any situation into a game. Why is this so? The most basic answer is that we love to compete, to test ourselves against arbitrary standards, our potential, or our peers. All we require is a reasonable chance of succeeding. Why do we keep at it if we are not doing well? We have an innate belief that with practice we will continually improve. In most of the above we compete less against other people than against our own abilities. This is a valuable source of true growth.

Contests such as spelling bees and the Maths Olympiads use this fundamental human characteristic to entice students into mastering skills or developing a way of thinking. Children (and sometimes the adults) may assume they are competing in order to win awards or to show they can do well on a national scale. The real purpose, however, is to get them to want to tackle richer, more demanding problems than those they may be used to. As with all education, the goal is growth. Adults should be careful to keep the students' sense of competition low-key. To maximise growth, the children should not feel pressured or threatened.

Students usually want two things out of any activity: to feel they belong on this level and to maintain a sense of progress.

To fill the first need and to satisfy most students, the Maths Olympiad contests are scaled so that the average student correctly solves about 40% of the problems during the year. Each contest includes problems specifically designed for both ends of the spectrum. Some problems are uncomplicated (even though they involve mathematical thinking) and their solutions can easily be understood by all. Such problems enable beginners to expect to solve their fair share of the problems within the time limits. Moreover, a student's scoring typically improves from year to year. Meanwhile, other problems challenge even the strongest mathletes. Success is quite attainable, but perfection is rare. In fact, less than one-third of one per cent of all participants correctly solve every problem for the year!

To fill the second need, all Olympiad problems involve higher order thought. Those who use guess-check-and-revise will be shown more elegant and efficient solutions when the problem is reviewed immediately afterwards. Even the best mathletes will learn new ways to apply familiar concepts inasmuch as the 25 problems may employ 40 or more principles, some in very subtle ways. The variety itself presents a challenge. Each time students learn something new, their sense of accomplishment is reinforced.

There are two completely separate divisions, **Division J** (for students up to 12 years of age and in school Year 6 or below) and **Division S** (for students up to 14 years of age and in school Year 8 or below). The Maths Olympiads consist of five monthly contests, between May to September inclusive. This provides constant reinforcement of mathematical thinking and problem solving, and helps to keep student interest high. Contests are conducted under strict standard testing conditions. All team members take the contests together. There are no make-ups. Each team can have as many as 30 students and a school can have as many teams as it wants; interested students should not be turned away.

Practices are important. Beginners develop a comfort level with contest problems and all mathletes improve their problem solving skills. (Towards this end APSMO provides 25 practice problems with detailed solutions prior to the first Olympiad.) Each solution also names the strategy employed; about half of the problems offer extensions to help the teacher develop each problem into a mini-lesson.

An individual's score is just the number of problems answered correctly. While in general work should be shown, it is not required on our contests. The team score, compiled after the last Olympiad, is simply the sum of the ten highest individual scores. Large teams and small teams are therefore on a fairly equal footing. No travelling is required; contests are held in the school.

Further, the generous awards structure enables the school to provide ample recognition. The awards package sent to each team includes a Certificate of Participation for every mathlete completing the Olympiad, encouragement awards, embroidered felt patches for about 50% of all mathletes, a variety of other awards for the high scorers on each team, and assorted team awards for about 25% of all teams. Further information is available at our website: *www.apsmo.info*.

11. Building a Program

So you want to start a maths team.

Any successful program requires solid support from many elements. Your students must be enthusiastic about the activity, their parents must see potential benefits for their children, and the school administration must feel they can justify the expenditure. Experienced Maths Olympiad coaches have used the following suggestions successfully. Few people do everything mentioned below, but the more you do, the more effective you are likely to be.

One way to build a program is to meet with the school principal and the director of mathematics to map out your procedures for devising a philosophy, recruiting students, building parental support, training students, conducting contests, and recognising accomplishments.

Set 1: Olympiad 1

1A
3 Minutes
79%

What number can replace the square to make the statement true?

$$5 \times 11 = \square + 12$$

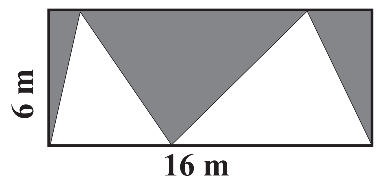
1B
6 Minutes
16%

The sum of the 3-digit number AAA and the 2-digit number BB is the 4-digit number CD6E. A, B, C, D, and E are different digits. What 4-digit number does CD6E represent?

$$\begin{array}{cccc} & A & A & A \\ + & & B & B \\ \hline C & D & 6 & E \end{array}$$

1C
7 Minutes
47%

What is the area, in square metres, of the shaded part of the rectangle?



1D
5 Minutes
6%

In simplest form, the fraction $\frac{60}{N}$ represents a whole number. N is also a whole number.
What is the total number of different values that N can be?

1E
5 Minutes
24%

A bowl contains 100 pieces of coloured lollies: 48 green, 30 red, 12 yellow, and 10 blue. They are all wrapped in foil, so you do not know the colour of any piece of candy. What is the least number of pieces you must take to be certain that you have at least 15 pieces of the same colour?

Hints: Division J

Set 1, Olympiad 1

1A. Multiply and then subtract.
1B. Examine A in the hundreds place. What addition produces the 2-digit sum CD?
1C. Split the rectangle into four smaller rectangles.
1D. Build an organised table of the pairs of factors of 60.
1E. How many pieces can you take before you are forced to take a 15^{th} piece of any one colour?

Set 1, Olympiad 2

2A. What could the tens digit be?
2B. How many 8-second periods are there in 2 minutes?
2C. What is the sum of the three numbers in each row, column, and diagonal?
2D. Think about each small square. What is its area? the length of each side?
2E. Count the number of 7s in an organised way.

Set 1, Olympiad 3

3A. How many weeks are there in 45 days?
3B. What is the sum of the five numbers?
3C. For every additional black square, how many more white squares are there?
3D. How is the area of the border related to the areas of the picture and the total area?
3E. Construct a chart of all possible outcomes. This is called a *sample space*.

Set 1, Olympiad 4

4A. What other numbers are said by the student who says, “1”?
4B. What is the lowest common denominator?
4C. Can the sums be equal if Mr. Sullivan lives in house number 2? 3? 4? 5?
4D. How many faces of each cube are painted?
4E. What is the lowest common multiple of 2, 3, 4, and 5? the next lowest common multiple?

Set 1, Olympiad 5

5A. Make a 4 by 4 table that compares each shirt colour to each name.
5B. To minimise the fraction, how large should the numerator be? the denominator?
5C. What was the area of the paper before the rectangles were cut out?
5D. What is the fewest number of small slices possible?
5E. In how many ways can you find three different counting numbers whose sum is 8?