

Revised & Expanded

# **Creative Problem Solving in School Mathematics**

**Australian Edition**

**Dr. George Lenchner**

**A Handbook For Teachers, Parents, Students, And Other  
Interested People.**

*Exploring Maths Through Problem Solving*



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## **Preface to Australian Edition**

Australasian Problem Solving Mathematical Olympiads (APSMO) Inc has been offering Mathematical Olympiads based on Dr Lenchner's model to schools throughout Australia, New Zealand and surrounding countries since 1987. The annual interschool Olympiads are held five times a year between May and September.

We take this opportunity to thank Dr Lenchner for his permission to reprint this revised and expanded version of his excellent text with modifications specific to Australian education.

This text is identical to Dr Lenchner's original text with the following modifications:

- Australian spelling.
- Changes in nomenclature such as imperial to decimal measurements, American coinage to Australian coinage. (Note : We have continued to use 1c and 2c coins although they are no longer in use in Australia).
- The sample questions remain true to the original, however, in certain situations they have been modified to reflect Australian standards. All care has been taken to ensure that the purpose and solution methods remain unchanged.

Thank you to Dr Anne Prescott, senior lecturer in primary and secondary mathematics education at the University of Technology, Sydney, for her valuable assistance in reviewing the alterations and ensuring that the text is correct and suitable for Australian students.

Jonathan Phegan  
Executive Director  
Australasian Problem Solving Mathematical Olympiads (APSMO) Inc

# Teaching Problem Solving

A

## Part A

1. What is Problem Solving?
2. Using a Four-Step Method
3. Choosing Problems
4. Evaluating Problems
5. Presenting Problems
6. Helping Students
7. Using Calculators and Computers

# Teaching Problem Solving

## 1. What is Problem Solving?

**A** It seems that everyone concerned with mathematics education today talks about problem solving. Professional organisations recommend that problem solving becomes the focus of school mathematics; curriculum guides list problem solving skills as key objectives at all levels; and it is difficult to find a meeting of educators that doesn't have at least one problem solving session on its agenda. However, we should be careful not to think of this interest in problem solving as just another "bandwagon." The ultimate goal of school mathematics at all times is to develop in our students the ability to solve problems.

Some teachers believe that the ability to solve problems develops automatically from mastery of computational skills. This is not necessarily true. Problem solving is itself a skill that needs to be taught, and mathematics teachers must make a special effort to do so.

Since we will be using the word "problem" repeatedly, let's begin by agreeing on its meaning. Any mathematical task can be classified as either an exercise or a problem. An **EXERCISE** is a task for which a procedure for solving is already known; frequently an exercise can be solved by the direct application of one or more computational procedures. A **PROBLEM** is more complex because the strategy for solving may not be immediately apparent; solving a problem requires some degree of creativity or originality on the part of the problem solver.

Let's look at an example: Suppose you are talking with your class about a collection of coins that consists of three 5c coins, two 10c coins, and one 20c coin. Before continuing, pause a moment to jot down some questions you might ask. Did you list any of the following?

1. How many coins are in the collection?
2. What is the total value of the collection in cents? in dollars?
3. Which of the sets of different types of coins has the greatest value? the least value?
4. How many different amounts of money can be made using one or more coins from this collection?
5. How many different combinations of one or more coins can be made using the coins in this collection?
6. How many other combinations of 5c, 10c, and 20c coins have the same value as the given collection?

*(Solutions are on page 132.)*

Notice that the first three questions listed have a quality different from the last three in that they can be solved by simple inspection or by using a computational algorithm. We consider the first three to be exercises. For the last three, no routine process of solving is applicable; the person faced with these questions must determine an appropriate strategy for solving before actually proceeding to solve. We classify these questions as problems.

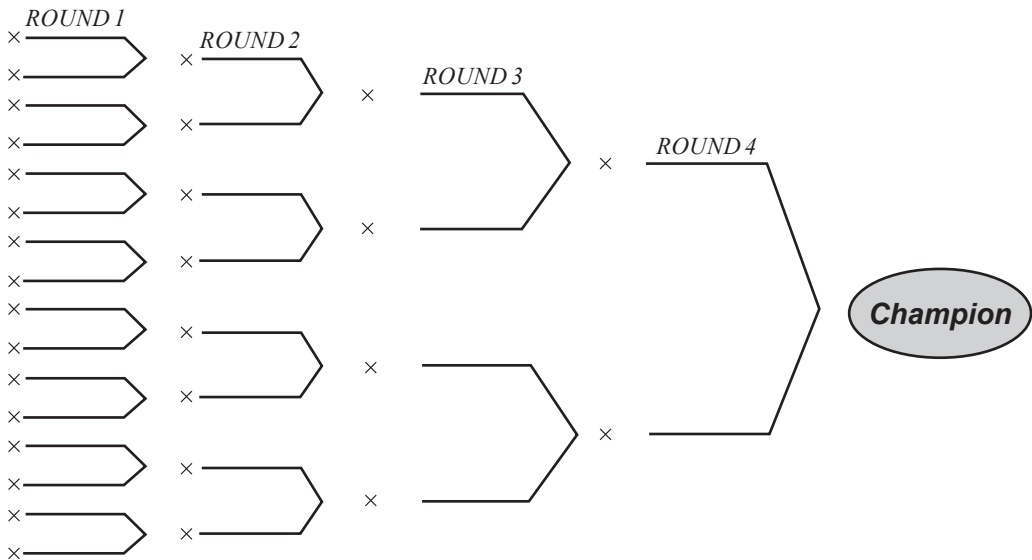
# Solutions to Part B Problems

If an answer and solution are given together, the answer itself is **boldfaced**, as in number 1 below.

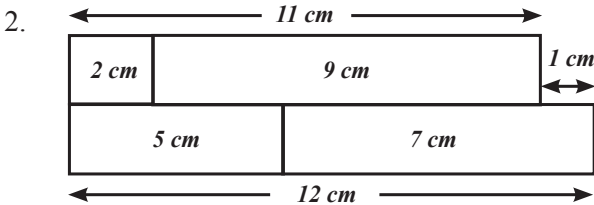
## 1. Drawing a Picture or Diagram

Pages 14-15

1. 4.



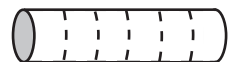
The championship team will have to play four tournament games.



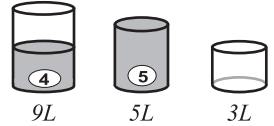
3. **20.** To obtain 4 pieces, the lumberjack needs to make only 3 cuts. Since 3 cuts take 12 minutes, each cut takes  $12 \div 3 = 4$  minutes.



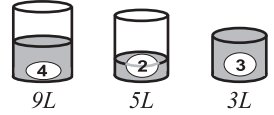
To obtain 6 pieces, the lumberjack will have to make 5 cuts. Since each cut takes 4 minutes, 5 cuts will take  $5 \times 4 = 20$  minutes.



4. (1) Fill the 9L container with water. Then empty as much of this water as possible into the 5L container, leaving 4 litres in the 9L container.



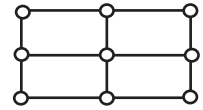
(2) Fill the 3L container from the 5L container, leaving 2 litres in the 5L container.



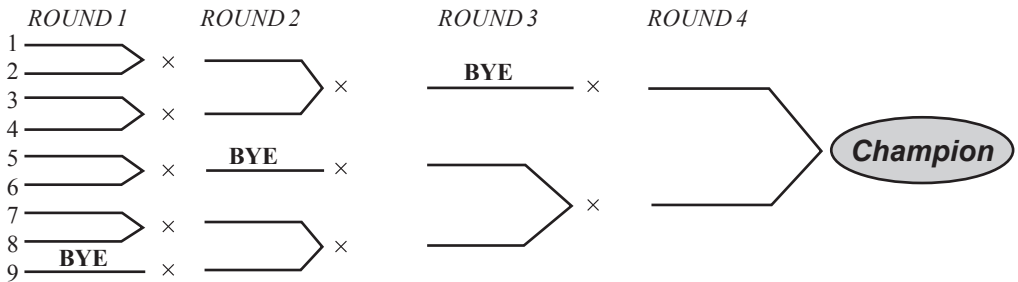
(3) Empty the 3 litres of water in the 3L container into the 9L container for a total of  $3 + 4 = 7$  litres.



5. 9. The least number of tacks you need is 9, as shown at the right.



6. 4. Refer to the layout of the tournament. For easy reference, competitors are numbered from 1 to 9.



Either player #7 or #8 would be the champion in this case. If any other player were the champion, that player would have played just 3 games. The maximum that a champion would have to play is 4 games.

## 2. Making an Organised List

Pages 16-17

1. 7.

3 Darts Hit Bull's Eye	2 Darts Hit Bull's Eye	1 Dart Hit Bull's Eye	0 Darts Hit Bull's Eye
$7 + 7 + 7 = 21$	$7 + 7 + 5 = 19$	$7 + 5 + 5 = 17$	$5 + 5 + 5 = 15$
	$7 + 7 + 3 = 17$	$7 + 5 + 3 = 15$	$5 + 5 + 3 = 13$
		$7 + 3 + 3 = 13$	$5 + 3 + 3 = 11$
			$3 + 3 + 3 = 9$

Seven different point totals are possible: 21, 19, 17, 15, 13, 11, and 9.



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*Boldfaced italicised listings indicate definitions.*

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